Solutions to the Attendance Quiz for Lecture 11

1. Find the eigenfunctions and eigenvalues for the following boundary value problem.

$$y'' + \lambda^2 y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(\pi) = 0 \quad .$$

Sol. The **general solution** of the DE is

$$y(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \quad .$$

For future reference

$$y'(x) = -c_1 \lambda \sin(\lambda x) + c_2 \lambda \cos(\lambda x) \quad .$$

Plug-in $x = 0$ into $y(x)$:

$$y(0) = c_1 \cos(\lambda \cdot 0) + c_2 \sin(\lambda \cdot 0) = c_1 \cos(0) + c_2 \sin(0) = c_1 \quad .$$

Since we are told that $y(0) = 0$, we get that $c_1 = 0$. Hence

$$y(x) = c_2 \sin(\lambda x) \quad .$$

Now

$$y'(x) = c_2 \lambda \cos(\lambda x) \quad .$$

Since we are told that $y'(\pi) = 0$, we get

$$y'(\pi) = c_2 \lambda \cos(\lambda \pi) = 0 \quad .$$

Since $c_2 \neq 0$ (or else we get that $y(x)$ is the zero function and that **does not count**) and also $\lambda \neq 0$, we **must** have

$$\cos(\lambda \pi) = 0 \quad .$$

We know from trig. that $\cos w$ is 0 when $w = \pi/2, 3\pi/2, 5\pi/2$ and in general $w = (n + \frac{1}{2})\pi$, it must be that

$$\lambda \pi = (n + \frac{1}{2})\pi \quad .$$

Solving for λ, (divide both sides by π) we get $\lambda = n + \frac{1}{2}$. So the **eigenvalues** are all "half-integers" $\pm 1/2, \pm 3/2, \ldots$, and in general $n + \frac{1}{2}$ (n integer), and the corresponding **eigenfunctions** are $\sin(n + \frac{1}{2})x$.

Ans.: The eigenvalues are $n + \frac{1}{2}$ (n any integer) and the respective eigenfunctions are $\sin(n + \frac{1}{2})x$.

Comments: **1.** About %30 of the people got it completely.

2. A fairly common error was replacing $\cos \lambda \pi$ by $(-1)^\lambda$. This is *wrong*! It is only true when λ happens to be an integer (whole numbers). If in doubt, plug in $\lambda = 1/2$ or whatever and see that you get something wrong.

3. Some people had trouble solving the equation

$$\cos(\lambda\pi) = 0 \quad .$$

Do it in **two** steps. First solve

$$\cos w = 0 \quad .$$

By the trig. identity $\cos(n + \frac{1}{2})\pi = 0$, the solutions are $w = (n + \frac{1}{2})\pi$.

But $w = \lambda\pi$ so

$$(n + \frac{1}{2})\pi = \lambda\pi \quad .$$

Now cancel out the π and get that the lucky λ are $\lambda = (n + \frac{1}{2})$.

4. Many people figured out correctly that $\lambda_n = n + \frac{1}{2}$ are the eigenvalues, but then said that the **eigenfunction** $y_n(x)$ equals $\cos(n + \frac{1}{2})x$, and some people said that it was $\lambda \cos(n + \frac{1}{2})x$. This is wrong! Don't plug the λ_n into $y'(x)$, but into $y(x)$ itself. Since $y(x) = c_2 \sin \lambda x$ it follows that $y_n(x) = c_2 \sin(n + \frac{1}{2})x$, and you can make $c_2 = 1$ to get the 'nicest-looking' eigenfunction.