Solutions to Dr. Z.'s Math 421, Exam #1

1. (15 points) Using the **definition** find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t \le 2; \\ -3, & \text{if } t \ge 2. \end{cases}$$

Sol.:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \; = \; \int_0^2 e^{-st}(1) \; + \; \int_2^\infty e^{-st}(-3)$$

$$= \frac{e^{-st}}{-s}\Big|_0^2 - 3\frac{e^{-st}}{-s}\Big|_2^\infty$$

$$-\frac{e^{-st}}{s}\Big|_0^2 + 3\frac{e^{-st}}{s}\Big|_2^\infty = -\left(\frac{e^{-s\cdot 2}}{s} - \frac{e^{-s\cdot 0}}{s}\right) + 3\left(\frac{e^{-s\cdot\infty}}{s} - \frac{e^{-s\cdot 2}}{s}\right)$$

$$= -\frac{e^{-2s}}{s} + \frac{1}{s} + 3 \cdot 0 - 3\frac{e^{-2s}}{s} = \frac{1}{s} - \frac{4e^{-2s}}{s}$$

Ans. to 1: $\frac{1}{s} - \frac{4e^{-2s}}{s}$.

2. (15 points) Find

$$\mathcal{L}^{-1}\{\frac{3s^2 - 6s + 2}{s(s-1)(s-2)}\}$$

Sol.: We use the **template**:

$$\frac{3s^2 - 6s + 2}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \quad .$$

Let's take the common denominator of the right side:

$$\frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{s(s-1)(s-2)} \quad .$$

The denominators automatically match, but to make the numerators agree we must have

$$3s^2 - 6s + 2 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

We now plug-in **convenient values**:

$s = 0$ gives

$$3 \cdot 0^2 - 6 \cdot 0 + 2 = A(0-1)(0-2) \quad ,$$

so

$$2 = 2A \quad ,$$

and we get that $A = 1$.

$s = 1$ gives

$$3 \cdot 1^2 - 6 \cdot 1 + 2 = B(1)(1-2) \quad ,$$

so

$$-1 = -B \quad ,$$

and we get that $B = 1$.

$s = 2$ gives

$$3 \cdot 2^2 - 6 \cdot 2 + 2 = C(2)(2-1) \quad ,$$

so

$$2 = 2C \quad ,$$

and we get that $C = 1$. Going back to the **template** we have:

$$\frac{3s^2 - 6s + 2}{s(s-1)(s-2)} = \frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-2} \quad .$$

Now it is time to apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{\frac{3s^2 - 6s + 2}{s(s-1)(s-2)}\} = \mathcal{L}^{-1}\{\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-2}\} = 1 + e^t + e^{2t} \quad .$$

Ans. to 2.: $1 + e^t + e^{2t}$.

Comments: Some people did it by equating coefficients of s^2, s, and s^0, and solving the system of three equations with three unknowns A, B, C. This is correct, but takes longer, and hence is more error-prone. Indeed quite a few of these people made careless errors. Some people made it even harder for themselves by writing $\frac{3s^2-6s+2}{s(s-1)(s-2)} = \frac{3s^2}{s(s-1)(s-2)} - \frac{6s}{s(s-1)(s-2)} + \frac{2}{s(s-1)(s-2)}$, and doing partial fraction decomposition for each of these fractions. While this is still correct (if you don't mess up!) It makes the problem much longer, by trading one problem with three problems.

3a. (7 points) Compute $\mathcal{L}\{(t+4)\mathcal{U}(t-4)\}$.

Sol. Obviously we have to use it, the formula $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$, but in order to use we first need some **preprocessing**

$$t + 4 = (t-4) + 8 \quad ,$$

so

$$\mathcal{L}\{(t+4)\mathcal{U}(t-4)\} = \mathcal{L}\{((t-4)+8)\mathcal{U}(t-4)\} = \mathcal{L}\{(t-4)\mathcal{U}(t-4)\} + \mathcal{L}\{8\mathcal{U}(t-4)\} = \frac{e^{-4s}}{s^2} + 8\frac{e^{-4s}}{s}$$

Ans. to 3a: $\frac{e^{-4s}}{s^2} + 8\frac{e^{-4s}}{s}$.

Comment: Another (a bit more sophisticated) way is to set $f(t-4) = t+4$, and write $x = t-4$ getting that $t = x+4$, so $f(x) = x+4+4 = x+8$ and going back to the t variable, we get

$f(t) = t+8$. Now use the formula $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$ and you get the same correct answer.

3b. (8 points) Compute

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{(s-2)^5}\} \quad .$$

Sol. We use the general formula

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \quad .$$

Here $a = 2$ and $F(s) = \frac{1}{(s-2)^5}$. By the tables, $f(t) = \frac{1}{4!}e^{2t}t^4 = \frac{1}{24}e^{2t}t^4$, so

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{(s-2)^5}\} = \frac{1}{24}e^{2(t-2)}(t-2)^4\mathcal{U}(t-2) \quad .$$

Sol. to 3b): $\frac{1}{24}e^{2(t-2)}(t-2)^4\mathcal{U}(t-2)$.

4. (15 points) Evaluate

$$\mathcal{L}\{\int_0^t \tau^5 e^{t-\tau}\, d\tau\} \quad .$$

Sol. The integral is the **convolution** $t^5 * e^t$. Using the formula

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} * \mathcal{L}\{g(t)\} \quad ,$$

we get

$$\mathcal{L}\{\int_0^t \tau^5 e^{t-\tau}\, d\tau\} = \mathcal{L}\{t^5 * e^t\} = \mathcal{L}\{t^5)\} * \mathcal{L}\{e^t\}$$

$$= \frac{5!}{s^6}\frac{1}{s-1} = \frac{120}{s^6(s-1)}$$

Ans. to 4: $\frac{120}{s^6(s-1)}$.

5. (15 points) Solve the initial-value problem

$$y'' + 2y' + y = \delta(t-3) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Sol. First apply $\mathcal{L}$:

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{\delta(t-3)\} \quad .$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = e^{-3s} \quad .$$

$$s^2Y - sy(0) - y'(0) + 2(sY - y(0)) + Y = e^{-3s} \quad .$$

$$s^2Y + 2sY + Y = e^{-3s} \quad .$$

$$(s^2 + 2s + 1)Y = e^{-3s} \quad .$$

$$(s+1)^2Y = e^{-3s} \quad .$$

So

$$Y = \frac{e^{-3s}}{(s+1)^2} \quad .$$

So

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{e^{-3s}}{(s+1)^2}\} \quad .$$

Now use the famous formula $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$, with $a = 3$ and $F(s) = \frac{1}{(s+1)^2}$ to get $f(t) = te^{-t}$ and finally

$$y(t) = \mathcal{L}^{-1}\{Y\} = (t-3)e^{-(t-3)}\mathcal{U}(t-3) \quad .$$

Ans. to 5.: $(t-3)e^{-(t-3)}\mathcal{U}(t-3)$.

6. Solve the system of ordinary differential equations with the given initial values

$$\frac{dx}{dt} = -x + y \quad ,$$

$$\frac{dy}{dt} = 2x \quad ,$$

subject to the initial conditions $x(0) = 0$, $y(0) = 2$.

Sol.: Apply $\mathcal{L}$:

$$\mathcal{L}\{\frac{dx}{dt}\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\} \quad ,$$

$$\mathcal{L}\{\frac{dy}{dt}\} = \mathcal{L}\{2x\} \quad .$$

So (as usuall, let $\mathcal{L}\{x\} = X$, $\mathcal{L}\{y\} = Y$)

$$sX - x(0) = -X + Y$$

$$sY - y(0) = 2X$$

So

$$sX - 0 = -X + Y$$

$$sY - 2 = 2X$$

Rearranging:

$$(s+1)X - Y = 0 \quad .$$

$$-2X + sY = 2 \quad .$$

It is easier to solve this system with back-substitution. From the first equation $Y = (s+1)X$. Plugging-in the second, we get:

$$-2X + s(s+1)X = 2 \quad .$$

Factor our X:

$$(-2 + s(s+1))X = 2 \quad .$$

$$(s^2+s-2)X = 2 \quad .$$

Factorize

$$(s+2)(s-1)X = 2 \quad .$$

Solve for X:

$$X = \frac{2}{(s+2)(s-1)} \quad .$$

Going back to Y:

$$Y = \frac{2s+2}{(s+2)(s-1)} \quad .$$

We now need to do partial-fraction decomposition for both X and Y. For X use the template

$$\frac{2}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \quad .$$

Common denominator

$$\frac{2}{(s+2)(s-1)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \quad .$$

Equating tops:

$$2 = A(s-1)+B(s+2) \quad .$$

Plugging in $s=1$ gives

$$2 = B(3) \quad .$$

So $B = \frac{2}{3}$.

Plugging in $s=-2$ gives

$$2 = A(-3) \quad .$$

So $A = -\frac{2}{3}$. Going back to the template:

$$X = -\frac{2}{3}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s-1} \quad .$$

For Y use the template

$$\frac{2s+2}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \quad .$$

Common denominator

$$\frac{2s+2}{(s+2)(s-1)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \quad .$$

Equating tops:

$$2s+2 = A(s-1)+B(s+2) \quad .$$

Plugging in $s=1$ gives

$$4 = B(3) \quad .$$

So $B = \frac{4}{3}$.

Plugging in $s = -2$ gives

$$-2 = A(-3) \quad .$$

So $A = \frac{2}{3}$. Going back to the template:

$$Y = \frac{2}{3}\frac{1}{s+2} + \frac{4}{3}\frac{1}{s-1} \quad .$$

Now it is time to apply $\mathcal{L}^{-1}$:

$$x(t) = \mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\{-\frac{2}{3}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s-1}\} = -\frac{2}{3}e^{-2t} + \frac{2}{3}e^{t} \quad .$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{\frac{2}{3}\frac{1}{s+2} + \frac{4}{3}\frac{1}{s-1}\} = \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t}$$

Ans. to 6.: $x(t) = -\frac{2}{3}e^{-2t} + \frac{2}{3}e^{t}$, $y(t) = \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t}$.

7. Find all the eigenvalues of the matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad ,$$

and determine a basis for each eigenspace.

Sol.: The **Characteristic matrix** is

$$\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \quad .$$

Taking its **determinant**, we have

$$\det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) - (1)(0) = (3-\lambda)(2-\lambda) \quad .$$

So the **characteristic equation** is:

$$(\lambda - 3)(\lambda - 2) = 0$$

So the **eigenvalues** are $\lambda = 3$ and $\lambda = 2$.

For each of them we must still find a basis for the **eigenspace**.

For $\lambda = 3$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 3 \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$3a + b = 3a \quad , \quad 2b = 3b \quad .$$

Solving we get $b = 0$ and a arbitrary. So a typical eigenvector is $\begin{bmatrix} a \\ 0 \end{bmatrix}$. Taking $a = 1$, we get that a basis for the eigenspace of the eigenvalue $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 2$, we have to find vector(s) $\begin{bmatrix} a \\ b \end{bmatrix}$, such that

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

In everyday notation:

$$3a + b = 2a \quad , \quad 2b = 2b \quad .$$

So

$$a + b = 0 \quad , \quad 0 = 0 \quad .$$

The second equation does not tell us anything we didn't know before, so $a = -b$, and a typical eigenvector is $\begin{bmatrix} -b \\ b \end{bmatrix}$. Taking $b = 1$, we get $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Ans. to 7 The eigenvalues are $\lambda = 3$ and a basis for its eigenspace is $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, and $\lambda = 2$ and a basis for its eigenspace is $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$.

Comment: There are more than one correct answer for the bases for the eigenspaces. Multiplying any of these vectors by a **non-zero** number still gives a correct answer.