MATH 421 (2), Dr. Z. , Solutions to the FINAL EXAM, Mon., Dec. 23, 2024 8:00-11:00 am, SEC 117

1. (15 pts.) Solve (from scratch!) the boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + 6u = 3\frac{\partial u}{\partial t} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , t > 0$$

$$u(x,0) = \sin 3x \quad , \quad 0 < x < \pi \quad .$$

Ans.: $u(x,t) = e^{-t} \sin 3x$

First we write

$$u(x,t) = X(x)T(t) \quad .$$

Plug this into the pde, to get

$$X''(x)T(t) + 6X(x)T(t) = 3X(x)T'(t) \quad .$$

Divide by $X(x)T(t)$:

$$\frac{X''(x)}{X(x)} + 6 = 3\frac{T'(t)}{T(t)} \quad .$$

It is convenient to move the 6 to the right, getting:

$$\frac{X''(x)}{X(x)} = 3\frac{T'(t)}{T(t)} - 6 \quad .$$

The left side does not depend on t and the right side does not depend on x. Since they are equal to each other neither of them depends on x or t, in other words, they are equal to the same constant. There are three cases, the constant is positive, zero, or negative. But if that constant is positive, there is no way that we get $u(x,0)$ being a trig. function, so we can assume that the constant is negative, and we write it as $-\lambda^2$. So we have two odes:

$$\frac{X''(x)}{X(x)} = -\lambda^2 \quad .$$

$$3\frac{T'(t)}{T(t)} - 6 = -\lambda^2 \quad .$$

Cleaning up:

$$X''(x) + \lambda^2 X(x) = 0$$

$$T'(t) - (2 - \lambda^2/3)T(t) = 0 \quad .$$

The general solution of the first equation is $c_1 \cos \lambda x + c_2 \sin \lambda x$. The general solution of the second one is $c_3 e^{(2-\lambda^2/3)t}$. So product solutions are $u(x,t) = (\cos \lambda x)(e^{(2-\lambda^2/3)t})$ and $u(x,t) = (\sin \lambda x)e^{(-2+\lambda^2/3)t}$. Since $u(0,t) = 0$ the first family is no good. So we are left with $u(x,t) = \sin \lambda x e^{(2-\lambda^2/3)t}$. Using $u(\pi,t) = 0$ gives $\sin(\lambda\pi) = 0$. Solving this trig. equation for λ gives $\lambda\pi = n\pi$ so $\lambda = n$ (n integer).
So far the candidates for building blocks are $u(x,t) = \sin(nx)e^{(2-n^2/3)t}$. By the principle of superposition any (finite or infinite)linear combination of them is a solution.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{(2-n^2/3)t} \quad .$$

So $u(x,0)$ gives a Fourier Sine expansion. In general given the initial function $u(x,0)$ we would find its Fourier Sine expansion and then stick $e^{(2-n^2/3)t}$ after the $\sin nx$ in the sigma. But this is a lucky case. $u(x,0)$ is a pure sine function, whose Fourier-Sine expansion equal itself! So $n = 3$ is the only term and we stick $e^{(2-3^2/3)t} = e^{-t}$ after the $\sin 3x$, getting that the answer is $(\sin 3x)e^{-t}$.

2. (15 pts.) Find the eigenvalues λ_n, and the corresponding eigenfunctions $y_n(x)$ for the following boundary value problem.

$$y'' + \lambda^2 y = 0 \quad , \quad y'(0) = 0 \quad , \quad y'(10) = 0 \quad .$$

Ans.: $\lambda_n = \frac{n\pi}{10}$ $y_n(x) = \cos(\frac{n\pi}{10}x)$,where n is an integer.

The general solution of the ode is

$$y(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \quad .$$

In order to take care of the boundary conditions, we need to first find $y'(x)$:

$$y'(x) = -\lambda c_1 \sin \lambda x + \lambda c_2 \cos \lambda x \quad .$$

So $y'(0) = \lambda c_2$. This means that $c_2 = 0$ and $y(x)$ must be of the form

$$y(x) = c_1 \cos \lambda x$$

and

$$y'(x) = -c_1 \lambda \sin \lambda x$$

Since $y'(10) = 0$ we need

$$y'(10) = -c_1 \lambda (\sin \lambda 10) \quad .$$

c_1 better not be zero, so we need to solve the trig. eq. $\sin(\lambda 10) = 0$. But the solution of $\sin w = 0$ is $w = n\pi$ (n integer), so we have $\lambda 10 = n\pi$. Solving for λ we get that the **eignevalues** are $\lambda_n = \frac{n\pi}{10}$. Going back to $y(x)$ (**NOT** $y'(x)$), we have (we can set $c_1 = 1$)

$$y_n(x) = \cos(\frac{n\pi}{10}x) \quad .$$

3. (15 pts.) Solve the pde

$$25u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0 \quad ,$$

subject to the **boundary-conditions**

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = \sin 4x \quad , \quad u_t(x,0) = 5\sin x + 10\sin 5x \quad , \quad 0 < x < \pi \quad .$$

Ans.:

$$u(x,t) = \sin 4x \cos 20t + \sin x \sin 5t + \frac{2}{5}\sin 5x \sin 25t \quad .$$

This is the **wave equation** with $a = 5$, with the usual (string-instrument in music) boundary conditions. Since both $u(x,0)$ $u_t(x,0)$ are either pure sine wave functions or finite combinations we can safely use Dr. Z.'s shortcut method.
To get $u(x,t)$ from $u(x,0)$ and $u_t(x,0)$, we multiply each $\sin nx$ term in $u(x,0)$ by $\cos(nat)$, and we multiply each $\sin nx$ term in $u_t(x,0)$ by $\frac{\sin(nat)}{na}$, and add them all up. So

$$u(x,t) = (\sin 4x)\cos(5\cdot 4t) + 5(\sin x)\frac{\cos(5t)}{5} + 10(\sin 5x)\frac{\sin(5\cdot 5t)}{5\cdot 5}$$

$$= \sin 4x \cos 20t + \sin x \sin 5t + \frac{2}{5}\sin 5x \sin 25t \quad .$$

4. (15 pts.) Find the half-range cosine expansion of $f(x) = x$ on $(0, 2\pi)$.

Ans.:

$$f(x) = \pi + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(\frac{nx}{2}) \quad .$$

OR (even better!)

$$\pi - \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(\frac{2k+1}{2}x) \quad .$$

Recall that the **first** step is to move from the interval $(0, L)$ to the interval $(0, \pi)$ by defining $g(x) = f(x\frac{L}{\pi})$. Here $L = 2\pi$ so

$$g(x) = f(x\frac{2\pi}{\pi}) = f(2x) = 2x \quad ,$$

on $(0, \pi)$. Recall that at the very end, once we would have the half-range cosine expansion of $g(x)$, we would go back to $f(x)$ using $f(x) = g(x/2)$.
Using the formula sheet

$$a_0 = \frac{2}{\pi}\int_0^{\pi} g(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (2x)\,dx = \frac{2}{\pi}x^2\Big|_0^{\pi} = \frac{2}{\pi}(\pi^2 - 0^2) = 2\pi$$

Next

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\,dx = \frac{4}{\pi}\int_0^{\pi} x\cos nx\,dx \quad .$$

From the formula sheet:

$$\int x\cos nx\,dx = \frac{\cos nx + nx\sin nx}{n^2} + C \quad .$$

So

$$a_n = \frac{4}{\pi}\int_0^{\pi} x\cos nx\,dx \quad = \frac{4}{\pi}\frac{\cos nx + nx\sin nx}{n^2}\Big|_0^{\pi} = \frac{4}{\pi}\left(\frac{\cos n\pi + n\pi\sin n\pi}{n^2} - \frac{\cos(n0) + n\pi\sin(0)}{n^2}\right) \quad .$$

Since $\sin n\pi = 0$, $\sin 0 =$, $\cos 0 = 1$ and $\cos n\pi = (-1)^n$ (since n is an integer) this becomes:

$$a_n = \frac{4}{\pi}\frac{(-1)^n - 1}{n^2} \quad .$$

From the formula sheet, the half-range cosine expansion of $g(x)$ (over $(0,\pi)$) is:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ,$$

so

$$g(x) = \pi + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx \quad .$$

To get back to $f(x)$, we use $f(x) = g(x/2)$ getting

$$f(x) = \pi + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(\frac{nx}{2}) \quad .$$

This is a **correct answer** that would give you full credit. However, an even better answer is to realize that when n is even $(-1)^n - 1$ is 0 and when n is odd it is always -2. So writing $n = 2k+1$ ($k = 0, 1, \ldots$), we get a better answer:

$$f(x) = \pi - \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(\frac{(2k+1)x}{2}) \quad .$$

5. (15 pts. altogether)
(a) (8 points) Show that the following set of two functions, over the given interval and weight function, is an orthogonal set.

$$\{ f_1(x) = 1, \quad f_2(x) = 5x - 3 \} \qquad [0,1] \quad , \quad w(x) = \sqrt{x} \quad .$$

(b) (7 points) Using **orthogonality** (no credit for other methods!) find numbers c_1 , c_2 such that

$$5x = c_1 f_1(x) + c_2 f_2(x) \quad .$$

Ans. to b): $c_1 = 3 \qquad c_2 = 1$

(a)

$$(f_1(x), f_2(x))_{w(x)} = \int_0^1 (1)(5x-3)\sqrt{x}\,dx = \int_0^1 (5x^{3/2} - 3x^{1/2})\,dx$$

$$= 5\frac{x^{5/2}}{5/2} - 3\frac{x^{3/2}}{3/2}\Big|_0^1 = (2x^{5/2} - 2x^{3/2})\Big|_0^1 = 2 - 2 = 0 \quad .$$

So they are orthogonal with respect to $w(x) = \sqrt{x}$ over the interval $[0,1]$.
(b)

$$c_1 = \frac{(f_1(x), f(x))_{w(x)}}{(f_1(x), f_1(x))_{w(x)}} = \frac{\int_0^1 (1)(5x)\sqrt{x}\,dx}{\int_0^1 (1)(1)\sqrt{x}\,dx} = \frac{\int_0^1 5x^{3/2}\,dx}{\int_0^1 x^{1/2}\,dx}$$

$$= \frac{2x^{5/2}\Big|_0^1}{(2/3)x^{3/2}\Big|_0^1} = \frac{(2-0)}{(2/3)(1-0)} = 3 \quad .$$

Now c_2 can be computed similarly, but it is rather tedious. By this stage, once we computed c_1 using orthogonality, it is OK to "cheat" and use simple algebra. Since

$$5x = (3)(1) + c_2(5x - 3) \quad ,$$

it is obvious that $c_2 = 1$ and it would have been foolish to do it the long way.

6. (15 pts.) Solve :

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad 0 < y < 1 \quad ,$$

Subject to

$$u(0,y) = 0 \quad , \quad u(\pi,y) = 0 \quad , \quad 0 < y < 1 \quad ;$$

$$u(x,0) = 0 \quad , \quad u(x,1) = (\sinh 4)\sin 4x + (\sinh 7)\sin 7x + (\sinh 10)\sin 10x \quad , \quad 0 < x < \pi \quad .$$

Ans.: $u(x,y) = \sin 4x \sinh 4y + \sin 7x \sinh 7y + \sin 10x \sinh 10y$

There are eight kinds of product solutions to Laplace's equation:

$$u(x,y) = \sin \lambda x \sinh \lambda y \quad , \quad u(x,y) = \sin \lambda x \cosh \lambda y \quad ,$$

$$u(x,y) = \cos \lambda x \sinh \lambda y \quad , \quad u(x,y) = \cos \lambda x \cosh \lambda y \quad ,$$

and the other ones obtained by trnasposing x and y. Since $u(0,y) = 0$ and $u(x,0)$ none of them survives except for $\sin \lambda x \sinh \lambda y$. Since $u(\pi,y) = 0$ we must have $\sin \lambda\pi = 0$ so $\lambda\pi = n\pi$ (n integer) so $\lambda = n$ (integer). So the building blocks for the pde plus the boundary conditions and the initial condition $u(x,0) = 0$ are

$$\sin nx \sinh ny \quad .$$

By the principle of superposition, any **linear combination** (finite or infinite)

$$u(x,y) = A_1 \sin x \sinh y + A_2 \sin 2x \sinh 2y + \ldots + A_n \sin nx \sinh n + \ldots$$

is yet another solution. Plugging-in $y = 1$ gives

$$u(x,1) = A_1 \sin x (\sinh 1) + A_2 \sin 2x (\sinh 2) + \ldots + A_n \sin nx (\sinh n) + \ldots$$

So we need to find the half-range sine series of $u(x,1)$ get the coefficients $A_1, A_2, \ldots$ and go back to $u(x,y)$. In this problem $u(x,1)$ is already a finite combination of pure sine waves (three of them) so it is already a sine-series. The n that show up are $n = 4$, $n = 7$ and $n = 10$, so it it obvious that

$$u(x,y) = \sin 4x \sinh 4y + \sin 7x \sinh 7y + \sin 10x \sinh 10y$$

7. (15 pts.) Find product solutions, if possible, to the partial differential equation

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0 \quad .$$

Ans.: $u(x,y) = Ce^{\frac{k}{2}x}e^{-\frac{k}{3}y}$ or $u(x,y) = Ce^{k(3x-2y)}$

Let

$$u(x,y) = X(x)Y(y) \quad .$$

Plug into the ode

$$2X'(x)Y(y) + 3X(x)Y'(y) = 0 \quad .$$

Divide by $X(x)Y(y)$:

$$\frac{2X'(x)Y(y) + 3X(x)Y'(y)}{X(x)Y(y)} = 0 \quad .$$

Simplify

$$2\frac{X'(x)}{X(x)} + 3\frac{Y'(y)}{Y(y)} = 0 \quad .$$

Leave the $X(x)$ stuff on the left and move the $Y(y)$ to the right:

$$2\frac{X'(x)}{X(x)} = -3\frac{Y'(y)}{Y(y)} \quad .$$

The left side does not depend on y, the right side does not depend on x. They are equal to each other, so **neither** depend on x or y, so they are both equal to the **same** constant, let's call it k. We have two odes:

$$2\frac{X'(x)}{X(x)} = k \quad ,$$

$$-3\frac{Y'(y)}{Y(y)} = k \quad .$$

Cleaning up

$$X'(x) - \frac{k}{2}X(x) = 0 \quad ,$$

$$Y'(y) + \frac{k}{3}Y(y) = 0 \quad ,$$

The general solutions are $X(x) = c_1e^{\frac{k}{2}x}$ $Y(y) = c_2e^{-\frac{k}{3}y}$, so $u(x,y) = c_1c_2e^{\frac{k}{2}x}e^{-\frac{k}{3}y}$. Putting $C = c_1c_2$ we get the answer $u(x,y) = Ce^{\frac{k}{2}x}e^{-\frac{k}{3}y}$. Replacing k by $6k$ and doing the algebra gives the nicer forms.

8. (15 pts.) Find

$$\mathcal{L}^{-1}\left\{ \frac{3s^2-1}{s^3-s} \right\}$$

Ans.:

$$1 + e^{-t} + e^{t} \quad .$$

We first factorize the denominator

$$\frac{3s^2-1}{s(s-1)(s+1)}$$

We next try **partial fraction decomposition** using the **template**

$$\frac{3s^2-1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} \quad .$$

Next we take common denominator of the right

$$\frac{3s^2-1}{s(s-1)(s+1)} = \frac{A(s+1)(s-1) + Bs(s-1) + Cs(s+1)}{s(s-1)(s+1)} \quad .$$

The bottoms automatically match, so we equate the tops

$$3s^2 - 1 = A(s+1)(s-1) + Bs(s-1) + Cs(s+1) \quad .$$

Convenient values: $s = 0$ gives $-1 = A(1)(-1)$ so $A = 1$; $s = 1$ gives $2 = C(1)(2)$ so $C = 1$; $s = -1$ gives $2 = B(-1)(-2)$ so $B = 1$. Going back to the template, we have:

$$\frac{3s^2-1}{s(s-1)(s+1)} = \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s-1} \quad .$$

Now, and only now, do we apply $\mathcal{L}^{-1}$:

$$\mathcal{L}^{-1}\{\frac{3s^2-1}{s(s-1)(s+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\} \quad .$$

And the answer follows from the table: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ and $\mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}$.

9. (15 pts.) **9a.** (7 points) Compute $\mathcal{L}\{(t+6)\mathcal{U}(t-6)\}$.

Ans.:

$$\frac{e^{-6s}}{s^2} + 12\frac{e^{-6s}}{s} \quad .$$

We first write (so that we can use the formula sheet formula $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = F(s)e^{-as}$)

$$\mathcal{L}\{(t+6)\mathcal{U}(t-6)\} = \mathcal{L}\{[(t-6)+12]\mathcal{U}(t-6)\} = \mathcal{L}\{(t-6)\mathcal{U}(t-6)\} + 12\mathcal{L}\{\mathcal{U}(t-6)\} \quad .$$

Using the formula with $f(t) = t$ and $f(t) = 1$ for the first and second term gives the answer.

9b. (8 points) Compute

$$\mathcal{L}^{-1}\{\frac{e^{-4s}}{(s+2)^3}\} \quad .$$

Ans.:

$$\frac{1}{2}(t-4)^2 e^{-2(t-4)}\mathcal{U}(t-4) \quad .$$

Here $F(s) = \frac{1}{(s+2)^3}$ $a = 4$ in the formula $\mathcal{L}^{-1}\{F(s)e^{-as}\} = f(t-a)\mathcal{U}(t-a)$. From the table $f(t) = \mathcal{L}^{-1}\{\frac{1}{(s+2)^3}\} = \frac{1}{2}t^2e^{-2t}$.

10. (15 pts.) Evaluate

$$\mathcal{L}\{\int_0^t \tau^{15} e^{3t-3\tau}\, d\tau\} \quad .$$

Ans.:

$$\frac{15!}{s^{16}(s-3)} \quad .$$

The integral is the **convolution** $t^{15} * e^{3t}$. Using the formula $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$, we have

$$\mathcal{L}\{t^{15} * e^{3t}\} = \mathcal{L}\{t^{15}\}\mathcal{L}\{e^{3t}\} = \frac{15!}{s^{16}} \cdot \frac{1}{s-3} = \frac{15!}{s^{16}(s-3)} \quad .$$

11. (15 pts.) Solve the initial-value problem

$$y'' + 6y' + 9y = \delta(t-1) \quad , \quad y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Ans.: $(t-1)e^{-3(t-1)}\mathcal{U}(t-1)$

We apply $\mathcal{L}$ to the ode, getting

$$\mathcal{L}\{y'' + 6y' + 9y\} = \mathcal{L}\{\delta(t-1)\}$$

Putting, as usuial $\mathcal{L}\{y(t)\} = Y(s)$,

$$s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 9Y(s) = e^{-s} \quad .$$

Since $y(0) = 0, y'(0) = 0$, this becomes

$$s^2Y(s) + 6sY(s) + 9Y(s) = e^{-s} \quad .$$

Factoring:

$$(s^2 + 6s + 9)Y(s) = e^{-s} \quad .$$

Solving for $Y(s)$:

$$Y(s) = \frac{e^{-s}}{s^2 + 6s + 9} = \frac{e^{-s}}{(s+3)^2} \quad .$$

So

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{e^{-s}}{(s+3)^2}\} \quad .$$

We use the formula $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$. Here $a = 1$ and $f(t) = \mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\} = te^{-3t}$ (from the table), so we get the answer.

12.(15 pts.) Using the Laplace Transform (no credit for other methods) solve the pde

$$u_{xx} = 4u_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0$$

subject to the **boundary-conditions**

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ,$$

and the **initial conditions**

$$u(x,0) = \sin(3x) \quad , \quad u_t(x,0) = 0 \quad , \quad 0 < x < \pi \quad ,$$

Ans.: $u(x,t) = sin(3x)\, cos(\frac{3}{2}t)$.

Let, as usual, $U(x,s) = \mathcal{L}(u(x,t))$, where the Laplce transform is w.r.t t and x is considered as a constant parameter.
Applying the Laplace transfrom to the pde

$$\mathcal{L}(u_{xx}) = 4\mathcal{L}(u_{tt})$$

gives

$$U''(x,s) = 4(s^2U(x,s) - su(x,0) - u_t(x,0)) \quad ,$$

giving

$$U''(x,s) = 4(s^2U(x,s) - s\, \sin 3x) \quad ,$$

subject to the **boundary conditions** $U(0,s) = 0$, $U(\pi,s) = 0$
So the ode in x is (suppresing the dependence on s for the sake of clarity):

$$U''(x) - 4s^2U(x) = -4s\sin 3x$$

The **homogeneous version**, $U''(x) - 4s^2U(x) = 0$, whose **characteristic equation** is $\lambda^2 - 4s^2 = 0$ giving the solutions $\lambda = 2s$ and $\lambda = -2s$. Hene the general solution of the hompgeneous version is

$$c_1e^{2sx} + c_2e^{-2sx} \quad .$$

Regarding a **particual solution**, we set

$$U(x,s) = A\sin 3x \quad ,$$

where the A is **to be determined** (comment: since the ode has no U' in it, it is safe not to do the more general template $U(x,s) = A\sin 3x + B\cos 3x$). Plugging it in into the above ode, we get

$$-9A\sin 3x - 4s^2\sin 3x = A(-4s\sin 3x) \quad .$$

Solving for A we have

$$A = \frac{4s}{9+4s^2} = \frac{s}{s^2 + (\frac{3}{2})^2} \quad .$$

Hence the general solution of the ode is

$$U(x,s) = c_1 e^{2sx} + c_2 e^{-2sx} + \frac{s}{s^2+(\frac{3}{2})^2}\sin 3x \quad .$$

pluging in $x = 0$ and $x = \pi$, we have

$$U(0,s) = c_1 + c_2 \quad .$$

$$U(\pi,s) = c_1 e^{2s\pi} + c_2 e^{-2s\pi} \quad .$$

So we have to solve the system

$$\{c_1 + c_2 = 0 \quad , \quad c_1 e^{2s\pi} + c_2 e^{-2s\pi} = 0\} \quad ,$$

whose solution is $c_1 = 0$ and $c_2 = 0$. Hence the solution of the ode is

$$U(x,s) = \frac{s}{s^2+(\frac{3}{2})^2}\sin 3x \quad .$$

Taking the **inverse Laplace transform** we have

$$u(x,t) = \mathcal{L}^{-1}(\frac{s}{s^2+(\frac{3}{2})^2}\sin 3x) = \sin 3x\cos(\frac{3}{2}t) \quad .$$

This is the answer.

13. (10 points) Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < 2 \quad , \quad 0 < y < 2 \quad ;$$

subject to the boundary conditions

$$u(0, y) = 1 \quad , \quad 0 < y < 2 \quad ; \quad u(2, y) = 4 \quad , \quad 0 < y < 2 \quad ;$$

$$u(x, 0) = 2 \quad , \quad 0 < x < 2 \quad ; \quad u(x, 2) = -2 \quad , \quad 0 < x < 2 \quad .$$

Ans.: The approximation of $u(1, 1)$ is: $\frac{5}{4}$

$$u(1,1) \tilde{} \frac{u(1,0) + u(2,1) + u(1,2) + u(0,1)}{4} = \frac{2 + 4 + (-2) + 1)}{4} = \frac{5}{4} \quad .$$

14. (10 pts.) Find all the eigenvalues of the matrix

$$\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} \quad ,$$

and determine a basis for each eigenspace.

Ans.: smaller eigenvalue: 1 corresponding eigenfunction: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

larger eigenvalue: 2 corresponding eigenfunction: $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$\det \begin{bmatrix} 10-\lambda & 12 \\ 4 & -7-\lambda \end{bmatrix} = (10-\lambda)(-7-\lambda)-(-6)(12) = (\lambda+7)(\lambda-10)+72 = \lambda^2-3\lambda+2 = (\lambda-1)(\lambda-2)$$

So the **characteristic equation** is

$$(\lambda-1)(\lambda-2) = 0 \quad .$$

Solving it: gives $\lambda = 1$ and $\lambda = 2$ as the two **eigenvalues**.
For each of these we need to find the corresponding eigenvectors.
When $\lambda = 1$ we have to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such

$$\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 1 \cdot \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

Doing the matrix-multiplication, we get two equations

$$10a - 6b = a \quad , \quad 12a - 7b = b \quad .$$

Cleaning-up

$$9a - 6b = 0 \quad , \quad 12a - 8b = 0 \quad .$$

But the second is a multiple of the first, so we can discard it, and get that the general solution is $b = 9a/6 = 3a/2$. Plugging this into the template $\begin{bmatrix} a \\ \frac{3}{2}a \end{bmatrix}$ Taking $a = 2$ (we can take any **non-zero** value for a) gives the eigenvector for $\lambda = 1$. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

When $\lambda = 2$ we have to find a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ such

$$\begin{bmatrix} 10 & -6 \\ 12 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 2 \cdot \begin{bmatrix} a \\ b \end{bmatrix} \quad .$$

Doing the matrix-multiplication, we get two equations

$$10a - 6b = 2a \quad , \quad 12a - 7b = 2b \quad .$$

Cleaning-up

$$8a - 6b = 0 \quad , \quad 10a - 8b = 0 \quad .$$

But the second is a multiple of the first, so we can discard it, and get that the general solution is $b = 8a/6 = 4a/3$. Plugging this into the template $\begin{bmatrix} a \\ \frac{4}{3}a \end{bmatrix}$ Taking $a = 3$ (we can take any **non-zero** value for a) gives the eigenvector for $\lambda = 2$. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.