Solutions to Dr. Z.'s Math 421(2) REAL Quiz #7

1. (5 points) Find the first two coefficients of the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 2, & \text{if } -1 < x < 0; \\ -1, & \text{if } 0 \le x < 1. \end{cases}$$

Sol. $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$. The general formula is

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \quad .$$

So:

$$c_0 = \frac{2 \cdot 0 + 1}{2} \int_{-1}^{1} f(x) P_0(x) = \frac{1}{2} \int_{-1}^{1} f(x) = \frac{1}{2} \int_{-1}^{0} 2 + \frac{1}{2} \int_{-1}^{1} (-1) = \frac{1}{2}(2-1) = \frac{1}{2}$$

$$c_1 = \frac{2 \cdot 1 + 1}{2} \int_{-1}^{1} f(x) P_1(x) = \frac{3}{2} \int_{-1}^{1} f(x) \cdot x = \frac{3}{2} \int_{-1}^{0} 2x + \frac{3}{2} \int_{0}^{1} (-1)(x)$$

$$= \frac{3}{2} x^2 \Big|_{-1}^{0} + \frac{3}{2}\left(\frac{-x^2}{2} \Big|_{0}^{1}\right) = \frac{3}{2}(0^2 - (-1)^2) + \frac{3}{2} \frac{-1-0}{2} = -\frac{3}{2} - \frac{3}{4} = -\frac{9}{4} \quad .$$

Ans. to 1: The first two coefficients are $c_0 = \frac{1}{2}$, $c_1 = -\frac{9}{4}$.

2. (5 points) Find product solutions, if possible, to the partial differential equation

$$\frac{\partial u}{\partial x} = 25 \frac{\partial u}{\partial y} \quad .$$

Sol. Try $u(x,y) = X(x)Y(y)$. So

$$X'(x)Y(y) = 25X(x)Y'(y) \quad .$$

Divided both sides by $X(x)Y(y)$:

$$\frac{X'(x)}{X(x)} = 25 \frac{Y'(y)}{Y(y)} \quad .$$

The left does not depend on y, and the right does not depend on x, and they are equal to each other, so **neither** depends on x or y, so they are both equal to a **number**, let's call it k. We have traded one pde with two odes:

$$\frac{X'(x)}{X(x} = k$$

$$25 \frac{Y'(y)}{Y(y)} = k \quad .$$

These are

$$X'(x) - kX(x) = 0 \quad ,$$

$$Y'(y) - (k/25)Y(y) = 0 \quad .$$

The general solutions are

$$X(x) = c_1 e^{kx} \quad ,$$

$$Y(y) = c_2 e^{(k/25)y} \quad .$$

So

$$u(x, y) = (c_1 e^{kx})(c_2 e^{(k/25)y}) = (c_1 c_2) e^{kx} e^{(k/25)y} \quad .$$

Renaming $c_1 c_2$, C, and simplifying we get

$$u(x, y) = C e^{kx+(k/25)y} \quad .$$

Ans. to 2: $u(x, y) = Ce^{kx+(k/25)y}$, where C is an arbitrary constant.