**Solutions to Dr. Z.'s Math 421(2), (Fall 2014, RU) REAL Quiz #4**

**1.** Show that the given functions are orthogonal on the given interval.

$$f_1(x) = e^{x^2}x^2 \quad , \quad f_2(x) = e^{-x^2}(x^3+2x) \quad , \quad [-1,1] \quad .$$

**Sol.**

$$(f_1, f_2) = \int_{-1}^{1} (e^{x^2}x^2)e^{-x^2}(x^3+2x)\,dx = \int_{-1}^{1} e^{x^2-x^2}x^2(x^3+2x)\,dx = \int_{-1}^{1} (x^5+2x^3)\,dx = 0 \quad ,$$

since the integrand $x^5+2x^3$ is an **odd function** and the integration is **symmetric**. Since $(f_1, f_2) = 0$ it means that the two functions $f_1(x), f_2(x)$ are orthogonal.

**Comment**: Almost everyone got it right!

**2.** Decide whether the set $\{\cos x\,,\, \cos 2x\,,\, \cos 3x\}$ is orthogonal over the interval $[0, \frac{\pi}{2}]$.

**Sol.**

$$(\cos x, \cos 2x) = \int_0^{\pi/2} \cos x \cos 2x\,dx \quad .$$

We have to use the famous trig. identity

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B)) \quad .$$

So

$$\cos x \cos 2x = \frac{1}{2}(\cos 3x + \cos(-x)) = \frac{1}{2}(\cos 3x + \cos x) \quad .$$

Integrating from 0 to $\pi/2$:

$$\int_0^{\pi/2} \cos x \cos 2x = \frac{1}{2}\int_0^{\pi/2} (\cos 3x + \cos x) = \frac{1}{2}\frac{\sin 3x}{3} + \frac{1}{2}\sin x$$

$$= \left(\frac{1}{6}\sin 3x + \frac{1}{2}\sin x\right)\Big|_0^{\pi/2} = \frac{1}{6}\sin\frac{3\pi}{2} - 0 + \frac{1}{2}\sin\frac{\pi}{2} - 0 \quad .$$

Remember that $\sin\frac{\pi}{2} = 1$ and $\sin\frac{3\pi}{2} = -1$, so this equals

$$\frac{1}{6}(-1) + \frac{1}{2}(1) = \frac{1}{3} \quad .$$

So the inner product $(\cos x, \cos 2x)$ is **not** zero, so these two functions are **not** orthogonal to each other, so there is no way that the whole family can be orthogonal, even if other pairs are OK.

**Sol. to 2**: The family is **not** orthogonal since we found two members, $\cos x$, and $\cos 2x$ that fail to be orthogonal to each other.

**Comment 1**: Quite a few people continued and found the inner products of the other two pairs, and found out that $(\cos x, \cos 3x)$ are orthogoanl and $(\cos 2x, \cos 3x)$ are not. This is **a waste of time**. To prove that a family is **orthogonal**, of course, you have to check every pair and make sure that the inner product is always zero, but in order to prove that a family of functions is **not** orthogonal, all you need is to come up with **one** pair whose inner-product is **not** zero.

To prove that a box of apples is a perfect, you need to check every apple. To prove that the box is **not** perfect, all you need is come up with **one** rotten apple.

**Comment 2**: About %40 of the people got it perfectly. Other peole got it partially.