Solution of Attendance Quiz for Lecture 20

1. Find the cosine and sine integral representation of the function $f(x) = e^{-2x}, x > 0$.

Sol. We have

$$A(\alpha) = 2\int_0^\infty f(x)\cos(\alpha x) \quad ,$$

$$B(\alpha) = 2\int_0^\infty f(x)\sin(\alpha x) \quad .$$

Let's do $A(\alpha)$ first, using the formula with $b = -2$ and $a = \alpha$:

$$A(\alpha) = 2\int_0^\infty e^{-2x}\cos(\alpha x)$$

$$= \frac{-4\cos(\alpha x)e^{-2x}}{\alpha^2+4} + \frac{2\alpha\sin(\alpha x)e^{-2x}}{a^2+4}\Big|_0^\infty$$

$$= \frac{-4\cos(\alpha\cdot\infty)e^{-2\cdot\infty}}{\alpha^2+4} - \frac{-4\cos(\alpha\cdot 0)e^{-2\cdot 0}}{\alpha^2+4} + \frac{2\alpha\sin(\alpha\cdot\infty)e^{-2\cdot\infty}}{\alpha^2+4} - \frac{2\alpha\sin(\alpha\cdot 0)e^{-2\cdot 0}}{\alpha^2+4}$$

$$= 0 + \frac{4}{\alpha^2+4} + 0 + 0 \quad .$$

$$= \frac{4}{\alpha^2+4} \quad .$$

(since $e^{-\infty} = 0$ and $\sin 0 = 0$). So

$$A(\alpha) = \frac{4}{\alpha^2+4} \quad .$$

Now it is time to do $B(\alpha)$:

$$B(\alpha) = 2\int_0^\infty e^{-2x}\sin(\alpha x) = \frac{-2\alpha e^{-2x}\cos(\alpha x)}{\alpha^2+4} + \frac{-4e^{-2x}\sin(\alpha x)}{\alpha^2+4}\Big|_0^\infty$$

$$= \frac{-2\alpha e^{-2\cdot\infty}\cos(\alpha\cdot\infty)}{\alpha^2+4} - \frac{-2\alpha e^{-2\cdot 0}\cos(\alpha\cdot 0)}{\alpha^2+4} + \frac{-4e^{-2\cdot\infty}\sin(\alpha\cdot\infty)}{\alpha^2+4} - + \frac{-4e^{-2\cdot 0}\sin(\alpha\cdot 0)}{\alpha^2+4}$$

$$= 0 + \frac{2\alpha}{\alpha^2+4} + 0 + 0$$

$$= \frac{2\alpha}{\alpha^2+4} \quad .$$

So

$$B(\alpha) = \frac{2\alpha}{\alpha^2+4} \quad .$$

Putting it into the **Fourier Cosine Integral Representation** Formula, we have:

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\alpha)\cos(\alpha x)\,d\alpha = \frac{1}{\pi}\int_0^\infty \frac{4}{\alpha^2+4}\cos(\alpha x)\,d\alpha = \frac{4}{\pi}\int_0^\infty \frac{1}{\alpha^2+4}\cos(\alpha x)\,d\alpha$$

And the Fourier Sine Integral Representation is:

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\alpha)\sin(\alpha x)\,d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\alpha}{\alpha^2+4}\sin(\alpha x)\,d\alpha \quad .$$