Solutions to the Attendance Quiz for Lecture 11

1. Find the eigenfunctions and eigenvalues for the following boundary value problem.

\[ y'' + \lambda^2 y = 0 \quad , \quad y(0) = 0 \quad , \quad y'(\pi) = 0 \ . \]

**Sol.** The general solution of the DE is

\[ y(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x) \ . \]

For future reference

\[ y'(x) = -c_1 \lambda \sin(\lambda x) + c_2 \lambda \cos(\lambda x) \ . \]

Plug-in \( x = 0 \) into \( y(x) \):

\[ y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 = 0 \ . \]

Since we are told that \( y(0) = 0 \), we get that \( c_1 = 0 \). Hence

\[ y(x) = c_2 \sin(\lambda x) \ . \]

Now

\[ y'(x) = c_2 \lambda \cos(\lambda x) \ . \]

Since we are told that \( y'(\pi) = 0 \), we get

\[ y'(\pi) = c_2 \lambda \cos(\lambda \pi) = 0 \ . \]

Since \( c_2 \neq 0 \) (or else we get that \( y(x) \) is the zero function and that does not count) and also \( \lambda \neq 0 \), we must have

\[ \cos(\lambda \pi) = 0 \ . \]

We know from trig. that \( \cos w = 0 \) when \( w = \pi/2, 3\pi/2, 5\pi/2 \) and in general \( w = (n + \frac{1}{2})\pi \), it must be that

\[ \lambda \pi = (n + \frac{1}{2})\pi \ . \]

Solving for \( \lambda \), (divide both sides by \( \pi \)) we get \( \lambda = n + \frac{1}{2} \). So the eigenvalues are all “half-integers” \( \pm 1/2, \pm 3/2, \ldots \), and in general \( n + \frac{1}{2} \) (\( n \) integer), and the corresponding eigenfunctions are \( \sin(n + \frac{1}{2})x \).

**Ans.:** The eigenvalues are \( n + \frac{1}{2} \) (\( n \) any integer) and the respective eigenfunctions are \( \sin(n + \frac{1}{2})x \).

**Comments:**

1. About %30 of the people got it completely.

2. A fairly common error was replacing \( \cos \lambda \pi \) by \( (-1)^\lambda \). This is wrong! It is only true when \( \lambda \) happens to be an integer (whole numbers). If in doubt, plug in \( \lambda = 1/2 \) or whatever and see that you get something wrong.
3. Some people had trouble solving the equation

$$\cos(\lambda \pi) = 0 .$$

Do it in **two** steps. First solve

$$\cos w = 0 .$$

By the trig. identity $\cos(n + \frac{1}{2})\pi = 0$, the solutions are $w = (n + \frac{1}{2})\pi$.

But $w = \lambda \pi$ so

$$(n + \frac{1}{2})\pi = \lambda \pi .$$

Now cancel out the $\pi$ and get that the lucky $\lambda$ are $\lambda = (n + \frac{1}{2})$.

4. Many people figured out correctly that $\lambda_n = n + \frac{1}{2}$ are the eigenvalues, but then said that the eigenfunction $y_n(x)$ equals $\cos(n + \frac{1}{2})x$, and some people said that it was $\lambda \cos(n + \frac{1}{2})x$. This is wrong! Don’t plug the $\lambda_n$ into $y'(x)$, but into $y(x)$ itself. Since $y(x) = c_2 \sin \lambda x$ it follows that $y_n(x) = c_2 \sin(n + \frac{1}{2})x$, and you can make $c_2 = 1$ to get the ‘nicest-looking’ eigenfunction.