Solutions to the Attendance Quiz for Lecture 10

1. (a) Find the complex Fourier series of $f(x) = 4x$ on the interval $-2\pi < x < 2\pi$.

Sol. of a: Here the interval is **not** $(-\pi, \pi)$ but $(-2\pi, 2\pi)$ so $p = 2\pi$, and we must transform it using $g(x) = f(xp/\pi) = f(x(2\pi)/\pi) = f(2x)$.

(For future reference, $f(x) = g(x/2)$).

So $g(x) = f(2x) = 4(2x) = 8x$, defined on the **fundamental interval** $(-\pi, \pi)$. So

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx}\, dx \quad , \quad n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots \quad .$$

In this problem $g(x) = 8x$, so

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} 8x e^{-inx}\, dx \quad = \frac{4}{\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \quad .$$

From the formula sheet: (if $c \neq 0$, or else you get nonsense).

$$\int x e^{cx}\, dx = \frac{-1 + cx}{c^2} e^{cx} + C \quad .$$

So

$$\int_{-\pi}^{\pi} x e^{-inx}\, dx \quad = \frac{-1 + (-in)x}{(-in)^2} e^{-inx} \Big|_{-\pi}^{\pi} \quad .$$

Simplifying, using that $i^2 = -1$, we get that this equals:

$$\frac{-1 - inx}{-n^2} e^{-inx} \Big|_{-\pi}^{\pi} = \frac{1 + inx}{n^2} e^{-inx} \Big|_{-\pi}^{\pi}$$

$$= \frac{1 + in(\pi)}{n^2} e^{-in(\pi)} - \frac{1 + in(-\pi)}{n^2} e^{-in(-\pi)} = \frac{1 + in\pi}{n^2} e^{-in\pi} - \frac{1 - in(\pi)}{n^2} e^{in(\pi)} \quad .$$

Using the famous identitites $e^{i\pi} = -1$ and $e^{-i\pi} = -1$, both $e^{in\pi}$ and $e^{-in\pi}$ equal $(-1)^n$ so the above equals:

$$\left(\frac{1 + in\pi}{n^2} - \frac{1 - in(\pi)}{n^2} \right) (-1)^n \quad .$$

$$\frac{2in\pi}{n^2} (-1)^n = \frac{2i(-1)^n \pi}{n} \quad .$$

Going back to c_n:

$$c_n = \frac{4}{\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \quad = \frac{8i(-1)^n}{n} \quad .$$

But this is only correct when $n \neq 0$. We need to treat c_0 separately:

$$c_0 = \frac{4}{\pi} \int_{-\pi}^{\pi} x\,dx \quad = \frac{4}{\pi} \frac{x^2}{2} \Big|_{-\pi}^{\pi} = \frac{4}{\pi} \frac{(-\pi)^2 - \pi^2}{2} = 0 \quad .$$

Going back to $g(x)$, we have

$$g(x) = \sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx} \quad .$$

Finally, we must go back to $f(x)$. $f(x) = g(x/2)$. Replacing x by $x/2$ right above gives

$$f(x) = \sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx/2} \quad .$$

Ans. to 1(a): The complex Fourier series of $f(x) = 4x$ on the interval $-2\pi < x < 2\pi$ is $\sum_{n=-\infty, n \neq 0}^{\infty} \frac{8i(-1)^n}{n} e^{inx/2}$.

(b) Find the Frequency Spectrum.

Sol.: Here $\omega = \pi/p = \frac{1}{2}$. Also

$$|c_n| = |\frac{8i(-1)^n}{n}| = \frac{8}{|n|} \quad (n \neq 0) \quad .$$

(Since $|i| = 1$ and $|(-1)^n| = 1$). Since the frequency spectrum is $(n\omega, |c_n|), n = 0, \pm 1, \pm 2,$, we have

$$\{(\frac{n}{2}, \frac{8}{|n|})\} \quad (n \neq 0) \quad , (0,0) \quad .$$

Ans. to 1(b): The frequency spectrum of $f(x) = 4x$ in $(-2\pi, 2\pi)$ is $\{(\frac{n}{2}, \frac{8}{|n|})\} \quad (n \neq 0)$ together with $(0,0)$.

Comments:

1. Only about %20 got both completely, but many came close, and would have done it if they had more time.

2. Another way is to use the formula for the general intevral $(-p,p)$.