Eigenvalues and Eigenvectors of 2×2 matrices

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Definition 0

A 2×2 square matrix $\mathbf{A}$ is a table of the form

$$\mathbf{A} := \left[\begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{array}\right], \tag{1}$$

such a matrix induces a function

$$f_{\mathbf{A}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

which takes vectors $\mathbf{x} \in \mathbb{R}^2$ as input and outputs a new vector $f_{\mathbf{A}}(\mathbf{x}) \in \mathbb{R}^2$. Let

$$\mathbf{x} := \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \tag{2}$$

the function $f_{\mathbf{A}}(\mathbf{x})$ induced by the matrix $\mathbf{A}$ is expressed by

$$f_{\mathbf{A}}(\mathbf{x}) := \mathbf{A} \cdot \mathbf{x} = \left[\begin{array}{c} a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 \\ a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 \end{array}\right] \tag{3}$$

Example 0

Let $\mathbf{A}$ denote a 2×2 matrix expressed by

$$\mathbf{A} = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] \tag{4}$$

and let $f_{\mathbf{A}}(\mathbf{x})$ denote the corresponding induced function of $\mathbf{x} := \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$ expressed by

$$f_{\mathbf{A}}(\mathbf{x}) := \mathbf{A} \cdot \mathbf{x} = \left[\begin{array}{c} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{array}\right] \tag{5}$$

incidentally $f\left(\left[\begin{array}{c} 5 \\ 6 \end{array}\right]\right)$ for instance is

$$f\left(\left[\begin{array}{c} 5 \\ 6 \end{array}\right]\right) = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] \cdot \left[\begin{array}{c} 5 \\ 6 \end{array}\right] = \left[\begin{array}{c} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{array}\right] = \left[\begin{array}{c} 17 \\ 39 \end{array}\right] \tag{6}$$

Definition 1

A nonzero vector $\mathbf{x}$ is an *eigenvector* of $\mathbf{A}$ if there exists a constant λ, called an *eigenvalue*, such that

$$f(\mathbf{x}) = \lambda\,\mathbf{x} \Leftrightarrow \mathbf{A}\cdot\mathbf{x} = \lambda\,\mathbf{x} \tag{7}$$

0.1 Solving for the Eigenvalues of 2×2 matrices

To obtain the eigenvalues for a 2×2 matrix $\mathbf{A}$ we solve the polynomial equation $det(\lambda I - A) = \mathbf{0}$:

$$\det\left(\begin{bmatrix} \lambda - a_{1,1} & -a_{1,2} \\ -a_{2,1} & \lambda - a_{2,2} \end{bmatrix}\right) = 0 \tag{8}$$

in other words we solve the *characteristic equation* expressed by

$$p_{\mathbf{A}}(\lambda) = (\lambda - a_{1,1})(\lambda - a_{2,2}) - a_{1,2}\,a_{2,1} = \lambda^2 - (a_{1,1} + a_{2,2})\,\lambda + (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) \tag{9}$$

Example 1 (part 1)

Let $\mathbf{A}$ denote the 2×2 matrix expressed by

$$\mathbf{A} = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix},$$

the corresponding eigenvalues are found by solving the equation

$$p(\lambda) = \det\left(\begin{bmatrix} \lambda - 0 & 6 \\ -1 & \lambda - 5 \end{bmatrix}\right) = 0$$
$$p(\lambda) = \lambda^2 - 5\,\lambda + 6$$

from which it follows that the eigenvalues are determined by solving the equation

$$\lambda^2 - 5\,\lambda + 6 = 0$$

whose solution is expressed by

$$\lambda \in \{2,\ 3\}$$

0.2 Solving for the Eigenvectors of 2×2 matrices

Once we have solved for the eigenvalues, the eigenvectors are computed by solving linear system of equations

$$f_{\mathbf{A}}(\mathbf{x}) = \lambda\mathbf{x} \Leftrightarrow \mathbf{A}\cdot\mathbf{x} = \lambda\,\mathbf{x} \tag{10}$$

for each one of the eigenvalues.

Example 1 (part 2)

We recall that in *Example part 1* we have determined that the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$$

are given by

$$\{\lambda_1 = 2, \quad \lambda_2 = 3\}\,.$$

The eigenvector corresponding to the eigenvalue $\lambda_1 = 2$ is given by solving the system

$$\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\,x_1 \\ 2\,x_2 \end{bmatrix}$$

equivalently written as

$$\begin{cases} -6\,x_2 & = & 2\,x_1 \\ x_1 + 5\,x_2 & = & 2x_2 \end{cases}$$

$$\Rightarrow \begin{cases} -2\,x_1 - 6\,x_2 & = & 0 \\ x_1 + 3\,x_2 & = & 0 \end{cases}$$

substitution or elimination yields

$$x_1 = -3\,x_2$$

hence the corresponding eigenvector space is determined by the parametric line $\forall s \in \mathbb{R}$

$$\begin{cases} x_1 & = & -3s \\ x_2 & = & s \end{cases}\,.$$

The eigenvector space associated with the eigenvalue $\lambda_1 = 2$ is the vector space specified by

$$s\,\mathbf{v}_1 = s \begin{bmatrix} -3 \\ 1 \end{bmatrix} \tag{11}$$

similarly we solve for the eigenvector corresponding to the eigenvalue $\lambda_2 = 3$

$$\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3\,x_1 \\ 3\,x_2 \end{bmatrix}$$

equivalently written as

$$\begin{cases} -6\,x_2 & = & 3\,x_1 \\ x_1 + 5\,x_2 & = & 3\,x_2 \end{cases}$$

$$\Rightarrow \begin{cases} -3x_1 - 6\,x_2 & = & 0 \\ x_1 + 2\,x_2 & = & 0 \end{cases}$$

substitution or elimination yields

$$x_1 = -2\,x_2$$

hence the corresponding eigenvectors space is determined by the parametric line $\forall t \in \mathbb{R}$

$$\left\{ \begin{array}{ccc} x_1 & = & -2t \\ x_2 & = & t \end{array} \right. .$$

The eigenvector space associated with the eigenvalue $\lambda_2 = 3$ is the vector space specified by

$$t\,\mathbf{v}_2 = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}. \tag{12}$$

In summary we say that the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$$

has eigenvalues/eigenvector pairs or equivalently we say that the spectrum of $\mathbf{A}$ denoted $\mathrm{Spec}(\mathbf{A})$ is expressed by

$$\mathrm{Spec}\,(\mathbf{A}) = \left\{ \left(\lambda_1 = 2,\ \mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right),\ \left(\lambda_2 = 3,\ \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \right\}$$