Dr. Z.'s Calc5 Lecture 9 Handout: Fourier Cosine and Sine Series

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Important Definitions

A function $f(x)$ is **even** if

$$f(-x) = f(x) \quad .$$

(Examples: $1, x^2, x^4, x^6, \ldots$, $\cos x$, e^{-x^2})

A function $f(x)$ is **odd** if

$$f(-x) = -f(x) \quad .$$

(Examples: $x, x^3, x^5, \ldots$, $\sin x$, xe^{-x^2})

Important Properties of Even/Odd Functions

(Even Function) $\times$ (Even Function) =Even Function

(Odd Function) $\times$ (Odd Function) =Even Function

(Even Function) $\times$ (Odd Function) =Odd Function

(Even Function) + (Even Function) =Even Function

(Odd Function) + (Odd Function) =Odd Function

Important Time Savers

If $f(x)$ is even then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

If $f(x)$ is odd then $\int_{-a}^{a} f(x)\,dx = 0$

Fourier Cosine Series (for Even Functions)

The Fourier series of an **even** function $f(x)$ on the interval $(-\pi, \pi)$ is the **cosine series** (no sines show up!)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ,$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx \quad ,$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx \quad ,$$

Fourier Sine Series (for Odd Functions)

The Fourier series of an **odd** function $f(x)$ on the interval $(-\pi, \pi)$ is the **sine series** (no cosines show up!)

$$\sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad ,$$

Half Range Expansion

If a function $f(x)$ is only defined on $(0, \pi)$, then we can extend it to $(-\pi, \pi)$ to either get an even function, and find its **cosine series**, or to an odd function and get its **sine series**. Both of them are supposed to converge to $f(x)$ in $(0, \pi)$.

General Note: For the sake of simplicity of formulas all the above are phrased in terms of the easiest intervals $(-\pi, \pi)$ and $(0, \pi)$. If you have intervals of the form $(-p, p)$ or $(0, p)$, consider $g(x) = f(xp/\pi)$, defined on $(-\pi, \pi)$ or $(0, \pi)$, find the answer for $g(x)$, and then go back and use $f(x) = g(x\pi/p)$.

Problem 9.1: Determine whether the following functions are (i) even (ii) odd (iii) neither.

a $f(x) = \sin 7x$

b $f(x) = \cos 2x$

c $f(x) = x^3 \cos 2x$

d $f(x) = 2x + x^3 + 7x^5$

e $f(x) = x + 3x^2 + x^3$

Solutions:

a: $f(-x) = \sin 7(-x) = \sin(-7x) = -\sin(7x) = -f(x)$; **odd** .

b: $f(-x) = \cos(2(-x)) = \cos(-2x) = \cos(2x) = f(x)$; **even** .

c: First Way:

$$f(-x) = (-x)^3 cos(2(-x)) = -x^3 \cos(-2x) = -x^3 \cos(2x) = -f(x) \quad .$$

So it is **odd**.

Second Way: cos is always **even**, x^3 is **odd** (since it is an **odd** power), even times odd is **odd**.

d: **odd**, since in $f(x)$ only **odd powers** show up.

e: **neither**. $f(x)$ has **both** even and odd powers showing up so it is neither.

Problem 9.2: Expand the given function in an appropriate cosine or sine series.

$$f(x) = \begin{cases} 1, & \text{if } -\pi < x < 0; \\ -1, & \text{if } 0 \le x < \pi. \end{cases}$$

Solution: It is clear that $f(x)$ is an **odd** function, so what we need is the **sine series**.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(-1)\sin nx\,dx = \frac{2}{\pi}\frac{(\cos nx)}{n}\Big|_0^{\pi}$$

$$= \frac{2}{\pi n}\left(\cos nx\Big|_0^{\pi}\right) = \frac{2}{\pi n}(\cos n\pi - cos0) = \frac{2}{\pi n}((-1)^n - 1) \quad .$$

So

$$f(x) = \sum_{n=1}^{\infty}\frac{2((-1)^n - 1)}{\pi n}\sin nx \quad ,$$

Ans. to 9.2: The Fourier-Sine series of $f(x)$ is $\sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{\pi n}\sin nx$.

Problem 9.3: Expand the given function in an appropriate cosine or sine series.

$$f(x) = \begin{cases} 1, & \text{if } -\pi < x < -\pi/2; \\ 0, & \text{if } -\pi/2 \le x < \pi/2; \\ 1, & \text{if } \pi/2 \le x < \pi; \end{cases}$$

Solution: It is clear that $f(x)$ is an **even** function, and the part that we care about, the lives in $(0,\pi)$ is:

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2; \\ 1, & \text{if } \pi/2 \le x < \pi \end{cases} \quad .$$

Since $f(x)$ came from an **even** function, we need is the **cosine series**.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\left(\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right)$$

$$= \frac{2}{\pi}\left(\int_0^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi} 1\,dx\right) = \frac{2}{\pi}\int_{\pi/2}^{\pi} 1\,dx = \frac{2}{\pi}\frac{\pi}{2} = 1 \quad .$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left(\int_0^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right)$$

$$= \frac{2}{\pi}\left(\int_0^{\pi/2} 0\cdot\cos nx\,dx + \int_{\pi/2}^{\pi} 1\cdot\cos nx\,dx\right) = \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos nx\,dx$$

$$= \frac{2}{\pi}\left(\frac{\sin nx}{n}\Big|_{\pi/2}^{\pi}\right) = \frac{2}{n\pi}(\sin n\pi - \sin n(\pi/2)) = \frac{2}{n\pi}(0 - \sin n(\pi/2)) = \frac{-2\sin n(\pi/2)}{\pi n}$$

So

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2\sin n\pi/2}{\pi n}\cos nx \quad ,$$

First Ans. to 9.3: The Fourier-cosine series of $f(x)$ is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2\sin n\pi/2}{\pi n}\cos nx \quad ,$$

Since $\sin n\pi/2 = 0$ when n is even and $\sin(2k+1)\pi/2 = (-1)^k$, we can simplify the above and write:

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{-2(-1)^k}{\pi(2k+1)}\cos(2k+1)x = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k+1)}\cos(2k+1)x \quad .$$

Better Ans. to 9.3: $\frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k+1)}\cos(2k+1)x$.

Problem 9.4: Find the half-range sine expansion of $f(x) = -1$ on $(0, \pi)$.

Solution to 9.4: We extend it to an **odd** function (since we were asked to get the **sine** expansion). Since it (the extension) is now an odd function (by constuction) the solution is **exactly the same** as Problem 9.2.

Problem 9.5: Find the half-range cosine expansion of the following function defined on $(0, \pi)$

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2; \\ 1, & \text{if } \pi/2 \le x < \pi; \end{cases}$$

Solution to 9.5: We extend it to an **even** function (since we were asked to get the **cosine** expansion). Since it (the extension) is now an even function (by constuction) the solution is **exactly the same** as Problem 9.3.

Problem 9.6: Find the half-range cosine expansion of the following function $g(x)$ defined on $(0, 10)$

$$g(x) = \begin{cases} 0, & \text{if } 0 \le x < 5; \\ 1, & \text{if } 5 \le x < 10 \end{cases} \quad .$$

Solution of 9.6: We transform the interval $(0, 10)$ to the usual interval $(0, \pi)$ by considering $f(x) = g(\frac{10}{\pi}x)$. so

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \pi/2; \\ 1, & \text{if } \pi/2 \le x < \pi \end{cases} \quad .$$

This is **exactly** problem 9.5, that, in turn, is **exactly** problem 9.3. So

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{\pi n}\cos nx \quad .$$

Finally, going back to the interval $(0, 10)$, using $g(x) = f(\frac{\pi}{10}x)$, all we have to do is replace x by $\frac{\pi}{10}x$, getting:

$$g(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{\pi n} \cos \frac{n\pi}{10} x \quad .$$