Dr. Z.'s Calc5 Lecture 21 Handout: Fourier Transform

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Important Definition: The Fourier Transform and The Inverse Fourier Transform

Fourier Transform:

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Transform:

$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha \;=\; f(x) \quad .$$

Important Definitions: The Fourier Sine Transform and The Inverse Fourier Sine Transform

Fourier Sine Transform:

$$\mathcal{F}_s\{f(x)\} = \int_{0}^{\infty} f(x)\sin\alpha x\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Sine Transform:

$$\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\sin\alpha x\,d\alpha \;=\; f(x) \quad .$$

Important Definitions: The Fourier Cosine Transform and The Inverse Fourier Cosine Transform

Fourier Cosine Transform:

$$\mathcal{F}_c\{f(x)\} = \int_{0}^{\infty} f(x)\cos\alpha x\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Cosine Transform:

$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_{0}^{\infty} F(\alpha)\cos\alpha x\,d\alpha \;=\; f(x) \quad .$$

Important property of the Fourier Transform

If $\mathcal{F}\{f(x)\} = F(\alpha)$ then for $n = 1, 2, 3, \ldots$.

$$\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha) \quad .$$

Important property of the Fourier Sine Transform

If $\mathcal{F}_s\{f(x)\} = F(\alpha)$ then

$$\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0) \quad .$$

Important property of the Fourier Cosine Transform

If $\mathcal{F}_c\{f(x)\} = F(\alpha)$ then

$$\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0) \quad .$$

Problem 21.1: Solve the heat equation $3u_{xx} = u_t, \quad -\infty < x < \infty \quad , \quad t > 0$ subject to

$$u(x,0) = \begin{cases} 4, & \text{if } |x| < 2; \\ 0 & \text{if } |x| > 2. \end{cases}$$

Solution of 21.1: We are looking for a function of the **two** variables x (space) and t(time), called $u(x,t)$, satisfying some conditions (a pde and initial condition). Instead we will first look for its Fourier transform, in the variable x, (leaving t alone) $\mathcal{F}\{u(x,t)\} = U(\alpha,t)$.

Applying $\mathcal{F}$ to the pde yields $\mathcal{F}\{3u_{xx}\} = \mathcal{F}\{u_t\}$. Using the property of $\mathcal{F}$ that $\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha)$, we get

$$3(i\alpha)^2 U(\alpha,t) = U(\alpha,t)_t$$

so

$$-3\alpha^2 U(\alpha,t) = \frac{d}{dt} U(\alpha,t) \quad .$$

We got the ode, in the variable t:

$$\frac{dU}{dt} + 3\alpha^2 U = 0 \quad .$$

Solving this simple ode, gives

$$U(\alpha,t) = ce^{-3\alpha^2 t} \quad .$$

where the constant c is $U(\alpha,0)$. $U(\alpha,0)$ is the Fourier transform of the function describing $u(x,0)$ namely of

$$f(x) = \begin{cases} 4, & \text{if } |x| < 2; \\ 0 & \text{if } |x| > 2. \end{cases}$$

The next task is to compute the Fourier transform of this $f(x)$.

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx = \int_{-2}^{2} 4e^{i\alpha x}\, dx = 4\frac{e^{i\alpha x}}{i\alpha}\Big|_{-2}^{2} = 4\frac{e^{i\alpha 2} - e^{i\alpha(-2)}}{i\alpha} = 8\frac{e^{2i\alpha} - e^{-2i\alpha}}{2i\alpha} = 8\frac{\sin 2\alpha}{\alpha}$$

So c above is $8\frac{\sin 2\alpha}{\alpha}$, and we get

$$U(\alpha,t) = 8\frac{\sin 2\alpha}{\alpha} e^{-3\alpha^2 t} \quad .$$

To go back to $u(x,t)$ we apply $\mathcal{F}^{-1}$

$$u(x,t) = \mathcal{F}^{-1}\{U(\alpha,t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\alpha,t)e^{-i\alpha x}\,d\alpha \;=\frac{4}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2\alpha}{\alpha}e^{-3\alpha^2 t}e^{-i\alpha x}\,d\alpha \quad .$$

This is a **correct** answer, **but**, using $e^{-i\alpha x} = \cos(\alpha x) - i\sin\alpha x$, and seeing that $\int_{-\infty}^{\infty}\frac{\sin 2\alpha}{\alpha}e^{-3\alpha^2 t}\sin\alpha x\,d\alpha = 0$, since the sine function is an **odd function**, we get the simpler solution (without any complex numbers!)

$$u(x,t) = \frac{4}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2\alpha\cos\alpha x}{\alpha}e^{-3\alpha^2 t}\,d\alpha \quad .$$

Ans. to 21.1: $u(x,t) = \frac{4}{\pi}\int_{-\infty}^{\infty}\frac{\sin 2\alpha\cos\alpha x}{\alpha}e^{-3\alpha^2 t}\,d\alpha$.

Problem 21.2: Solve the pde

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < \pi \quad , \quad y > 0$$

subject to the boundary conditions

$$u(0,y) = 0 \quad , \quad u(\pi,y) = 3e^{-2y} \quad , \quad y > 0 \quad ;$$

$$u_y(x,0) = 0 \quad , \quad 0 < x < \pi \quad .$$

Solution: Instead of looking for $u(x,t)$ we will look for its Fourier cosine transform with respect to the variable y, leaving x alone, $\mathcal{F}_c\{u(x,y)\} = U(x,\alpha)$. Applying $\mathcal{F}_c$ to the pde gives

$$\mathcal{F}_c\{u_{xx}\} + \mathcal{F}_c\{u_{yy}\} = \mathcal{F}_c\{0\} \quad ,0 < x < \pi \quad .$$

Since $\mathcal{F}_c\{u_{yy}\} = -\alpha^2 U(x,\alpha) - u_y(x,0)$, and the last boundary condition says that $u_y(x,0) = 0$, we have $\mathcal{F}_c\{u_{yy}\} = -\alpha^2 U(x,\alpha)$. Of course $\mathcal{F}_c\{u_{xx}\} = U_{xx}(x,\alpha)$. So

$$\frac{d^2U}{dx^2} - \alpha^2 U = 0 \quad , \quad 0 < x < \pi \quad .$$

The general solution of this ode is

$$U(x,\alpha) = c_1\cosh\alpha x + c_2\sinh\alpha x \quad .$$

We need to translate the **boundary conditions** $u(0,y) = 0, u(\pi,y) = 3e^{-2y}$ from the u-language to the U-language, by applying $\mathcal{F}_c$.

$$U(0,\alpha) = \mathcal{F}_c(0) = 0 \quad , \quad U(\pi,\alpha) = 3\mathcal{F}_c\{e^{-2y}\} \quad .$$

Now, from Maple, or from a table of integrals:

$$U(\pi,\alpha) = \mathcal{F}_c\{3e^{-2y}\} = \int_0^{\infty} 3e^{-2y}\cos\alpha y\,dy \;= \frac{6}{4+\alpha^2} \quad .$$

We have the system of two equations and two unknowns

$$U(0,\alpha) = c_1 \cosh 0 + c_2 \sinh 0 \quad ,$$

$$U(\pi,\alpha) = c_1 \cosh \alpha\pi + c_2 \sinh \alpha\pi \quad .$$

So

$$0 = c_1 \quad ,$$

$$\frac{6}{4+\alpha^2} = c_1 \cosh \alpha\pi + c_2 \sinh \alpha\pi \quad .$$

and we get $c_1 = 0$ and $c_2 = \frac{6}{(4+\alpha^2)\sinh \alpha\pi}$, establishing that

$$U(x,\alpha) = \frac{6 \sinh \alpha x}{(4+\alpha^2)\sinh \alpha\pi} \quad .$$

Applying $\mathcal{F}_c^{-1}$ we get

$$u(x,y) = \frac{12}{\pi} \int_0^\infty \frac{\sinh \alpha x}{(4+\alpha^2)\sinh \alpha\pi} \cos \alpha y \, d\alpha \quad .$$

Ans. to 21.2: $u(x,y) = \frac{12}{\pi} \int_0^\infty \frac{\sinh \alpha x}{(4+\alpha^2)\sinh \alpha\pi} \cos \alpha y \, d\alpha \quad .$

Note: You have to be flexible! Here the "active" variable was y, not the usual x, so the formulas for $\mathcal{F}_c$ and $\mathcal{F}_c^{-1}$ had to be adjusted accordingly.

Note: If the boundary condition would have given $u(x,0)$ rather than $u_y(x,0)$ then one should use the Sine Fourier Transform.