Dr. Z.'s Calc5 Lecture 14 Handout

By Doron Zeilberger

The Three Most Important PDEs

Heat Equation: (Parabolic Type)

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0.$$

(where k is a **positive** number) .

Wave Equation (Hyperbolic Type)

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad , \quad 0 < x < L \quad , \quad t > 0.$$

Laplace's Equation (Elliptic Type)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad .$$

Types of Initial Conditions

For the Heat Equation: $u(x,0) = f(x)$, where $f(x)$ is the temperature at point x on the rod $[0, L]$ at the **beginning** ($t = 0$).

For the Wave Equation: $u(x,0) = f(x)$, where $f(x)$ is how the stretched string looks at the **beginning** ($t = 0$), and $u_t(x,0) = g(x)$, where $g(x)$ is its **velocity** function at the very beginning.

For Laplace's Equation: **None**, $u(x,y)$ is a function of (x,y) **not** of time, t.

Types of Boundary Conditions:

For the Heat Equation:

$u(0,t) = u_0$, if the **left-side** is kept at constant temperature u_0.

$u(L,t) = u_0$, if the **right-side** is kept at constant temperature u_0.

$u_x(0,t) = 0$, if the **left-side** is **insulated**.

$u_x(L,t) = 0$, if the **right-side** is **insulated**.

Wave Equation:

$u(0,t) = 0$, $u(L,t) = 0$, $t > 0$, if **both** ends of the strings are **secured** to the x-axis.

$u(0,t) = f(t)$, $t > 0$ if the left side has **transversal motion** of the form $f(t)$.

$u(L,t) = g(t)$, $t > 0$ if the right side has **transversal motion** of the form $g(t)$.

Laplace's Equation: If $u(x,y)$ is defined in $0 < x < a, 0 < y < b$, then the following types may show up (but **not** all of them at once)

$$u(0,y) = f(y) \quad , \quad 0 < y < b \quad (left \quad side \quad held \quad at \quad temperature \quad f(y))$$
$$u(a,y) = g(y) \quad , \quad 0 < y < b \quad (right \quad side \quad held \quad at \quad temperature \quad g(y))$$
$$u(x,0) = F(x) \quad , \quad 0 < x < a \quad (bottom \quad side \quad held \quad at \quad temperature \quad F(x))$$
$$u(x,b) = G(x) \quad , \quad 0 < x < a \quad (top \quad side \quad held \quad at \quad temperature \quad G(x))$$

$$u_x(0,y) = 0 \quad , \quad 0 < y < b \quad (left \quad side \quad is \quad insulated)$$
$$u_x(a,y) = 0 \quad , \quad 0 < y < b \quad (right \quad side \quad is \quad insulated)$$
$$u_y(x,0) = 0 \quad , \quad 0 < x < a \quad (bottom \quad side \quad is \quad insulated)$$
$$u_y(x,b) = 0 \quad , \quad 0 < x < a \quad (top \quad side \quad is \quad insulated)$$

Problem 14.1: A rod of length L coincides with the interval $[0,L]$ on the x-axis. Set up the boundary value problem for the temperature $u(x,t)$.

a. The left end is insulated, the right-end is held at temperature 100, the initial temperature is $f(x)$.

b. The left end is held at temperature u_0, the right end is insulated and initial temperature is 0 throughout.

Solution: The **pde** for **temperature of rods** $u(x,t)$ is **always**

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad k > 0 \quad ,0 < x < L \quad , \quad t > 0.$$

Only the **initial** and **boundary** conditions change from one problem to the next.

For **a.**: $u_x(0,t) = 0$ (since the left end is insulated), $u(L,t) = 100$, $u(x,0) = f(x)$.

For **b.**: $u(0,t) = u_0$ (since the left end is always at temperature u_0), $u_x(L,t) = 0$ (since the right end is insulated), $u(x,0) = 0$.

Problem 14.2: A string of length L coincides with the interval $[0,L]$ on the x-axis. Set up the boundary-value problem for the displacement $u(x,t)$.

a. The ends are secured to the x-axis. The string is released from rest from the initial displacement $x^2(L-x)^2$.

b. The ends are secured to the x-axis. The string is along the x-axis at the very beginning, but has initial velocity $sin(\pi x/L)$.

c. The right end is secured to the x-axis, but the left end moves in a **transversal** manner according to $\sin(3\pi t)$. Initially the string is undisplaced and is at rest.

Solution. For displacement of a **string** the **pde** is **always** the Wave Equation:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad ,0 < x < L \quad , \quad t > 0.$$

Only the initial and boundary conditions change.

For **a**: $u(0,t) = 0, u(L,t) = 0$ (since both ends are secured); $u(x,0) = x^2(L-x)^2$ (that's how it looks at the start), $u_t(x,0) = 0$ (since it starts at rest).

For **b**: $u(0,t) = 0, u(L,t) = 0$ (since both ends are secured); $u(x,0) = 0$,$u_t(x,0) = \sin(\pi x/L)$,

For **c**: $u(0,t) = \sin(3\pi t)$ (that's how it moves at $x = 0$) $u(L,t) = 0$ (the right end is secured), $u(x,0) = 0$ (since initially it is **undisplaced**, $u_t(x,0) = 0$ (since initially it is **at rest**).

Problem 14.3: Set up the boundary value problem for the steady-state temperature $u(x,y)$, where a thin rectangular plate coincides with the region in the xy-plane defined by $0 \le x \le 10, 0 \le y \le 20$. The left end and the bottom of the plate are insulated, the top of the plate is held at temperature 50, and the right end of the plate is held at temperature $g(y)$.

Solution: The **pde** for "*steady-state temperature*" (in the plane) is **always** Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad ,$$

for the appropriate a and b. In this problem $a = 10, b = 20$, so the pde is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < 10 \quad , \quad 0 < y < 20 \quad ,$$

Since the left end ($x = 0, 0 < y < 20$) is **insulated** we have: $u_x(0,y) = 0$.

Since the bottm end ($0 < x < 10, y = 0$) is **insulated** we have: $u_y(x,0) = 0$.

Since the top of the plate ($0 < x < 10, y = 20$), is held at temperature 50, we have $u(x,20) = 50$.

Since the right end of the plate ($x = 10, 0 < y < 20$), is held at temperature $g(y)$, we have $u(10,y) = g(y)$.

Answer to 14.3:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < 10 \quad , \quad 0 < y < 20 \quad ,$$

subject to the **boundary conditions**

$$u_x(0,y) = 0 \quad , \quad 0 < y < 20 \quad ; \quad u(10,y) = g(y) \quad , \quad 0 < y < 20 \quad ;$$

$$u_y(x,0) = 0 \quad , \quad 0 < x < 10 \quad ; \quad u(x,20) = 50 \quad , \quad 0 < x < 10 \quad .$$