1. (5 points) Solve the boundary value pde problem:
\[ u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0; \]
\[ u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0; \]
\[ u(x, 0) = \sin(7\pi x), \quad u_t(x, 0) = \sin(8\pi x), \quad 0 < x < \pi. \]

**Sol.** Note, this is a trick question! The “shortcut” method is not applicable, since \( \sin(7\pi x) \) and \( \sin(8\pi x) \) are not of the form \( \sin(nx) \) for integer \( n \). Nevertheless, I gave three out of five points, to people who “used” the “shortcut method” (wrongly of course, since it is not applicable in this case), since I intended the initial conditions to be \( u(x, 0) = \sin(7x) \), \( u_t(x, 0) = \sin(8x) \), and it was really a typo. For this problem the answer is simply (here \( a = 1 \))
\[ u(x, t) = \sin(7x) \cos(7t) + \frac{\sin(8x) \sin(8t)}{8}. \]

But the problem, as given still makes sense, except that it takes too long to do in five minutes. People who wrote
\[ u(x, t) = \sum_{n=1}^{\infty} A_n \sin(nx) \cos(nt) + \sum_{n=1}^{\infty} B_n \sin(nx) \sin(nt), \]
where
\[ A_n = \frac{2}{\pi} \int_0^\pi \sin(7\pi x) \sin nx \, dx, \quad B_n = \frac{2}{n\pi} \int_0^\pi \sin(8\pi x) \sin nx \, dx \]
without actually computing these complicated and awkward integrals got full credit. (If you really want to do it, you use the product formula for \( \sin A \sin B \) from the formula sheet, and then integrate the trig functions, getting an ugly mess for \( A_n \) and \( B_n \).)

2. (5 points) Solve:
\[ u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \]
Subject to
\[ u(0, y) = 0, \quad u(\pi, y) = 0, \quad 0 < y < 1; \]
\[ u(x, 0) = 0, \quad u(x, 1) = 5, \quad 0 < x < \pi. \]

**Sol.** There are eight families of product solutions
\[ \cos(\lambda x) \cosh(\lambda y), \quad \cos(\lambda x) \sinh(\lambda y), \quad \sin(\lambda x) \cosh(\lambda y), \quad \sin(\lambda x) \sinh(\lambda y), \]
and four others obtained by exchanging \( \cos \) and \( \cosh \) and \( \sin \) and \( \sinh \). To make \( u(0, y) = 0 \) happy, we must rule out the first two, since plugging-in \( x = 0 \) does not yield 0. To make \( u(x, 0) = 0 \) happy, we must rule out the third one, so all that remains is \( u(x, y) = \sin(\lambda x) \sinh(\lambda y) \) (and its analog
\( u(x, y) = \sinh(\lambda x) \sin(\lambda y) \). To make \( u(\pi, y) = 0 \) we need \( u(\pi, y) = \sin(\lambda \pi) \sinh(\lambda y) = 0 \). Since \( \sinh(\lambda y) \) is not the zero function, we must insist that \( \sin(\lambda \pi) = 0 \) (for the analog \( \sinh(\lambda \pi) = 0 \), but this never happens, so we can forget about that option).

Solving the trig equation \( \sin(\lambda \pi) = 0 \) yields \( \lambda \pi = n\pi \), solving for \( \lambda \) gives \( \lambda = n \), \( n \) integer.

So the infinite family \( \{ (\sin nx)(\sinh ny) \} \) are all solutions of the pde plus the first three conditions. By the superposition principle so is any combination

\[
  u(x, y) = \sum_{n=1}^{\infty} A_n (\sin nx)(\sinh ny) ,
\]

for any numbers \( A_1, A_2, A_3, \ldots \).

It remains to make the last condition happy \( u(x, 1) = 5 \). Plugging in \( y = 1 \):

\[
  u(x, 1) = \sum_{n=1}^{\infty} A_n (\sin nx)(\sinh n) = \sum_{n=1}^{\infty} (A_n \sinh n)(\sin nx)
\]

So

\[
  5 = \sum_{n=1}^{\infty} (A_n \sinh n)(\sin nx) \, .
\]

So \( A_n \sinh n \) are the Fourier-Sine coefficients of 5.

We have to find the Fourier-Sine series of 5 (no shortcuts are possible!) By the formula

\[
  f(x) = \sum_{n=1}^{\infty} a_n \sin nx
\]

with

\[
  a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx ,
\]

we get

\[
  a_n = \frac{2}{\pi} \int_0^{\pi} 5 \sin nx \, dx = \frac{10}{\pi} \int_0^{\pi} \sin nx \, dx = \left( \frac{10}{\pi} \right) \left( -\cos nx \right) \bigg|_0^{\pi} = -10 \frac{n}{\pi} (\cos 0 - \cos n\pi) = -10 \frac{n}{\pi} (1 - (-1)^n) \, ,
\]

so

\[
  A_n \sinh n = \frac{10}{n\pi}((-1)^n - 1) \, ,
\]

and solving for \( A_n \) we get

\[
  A_n = \frac{10}{n\pi \sinh(n)}((-1)^n - 1).\]

Going back to \( u(x, y) \) we get

\[
  u(x, y) = \sum_{n=1}^{\infty} \frac{10(1 - (-1)^n)}{n\sinh(n)\pi} (\sin nx)(\sinh ny) \, .
\]

This is the answer.

Comments: 1. This was a long problem, so people who wrote

\[
  A_n = \frac{10}{\pi \sinh(n)} \int_0^{\pi} \sin nx \, dx ,
\]

also got full credit.