1. (5 points) Find the first two coefficients of the Fourier-Legendre expansion of $f(x) = \begin{cases} 2, & \text{if } -1 < x < 0; \\ -1, & \text{if } 0 \leq x < 1. \end{cases}$

**Sol.** $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$. The general formula is

$$c_n = \frac{2n + 1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx.$$  

So:

$$c_0 = \frac{2 \cdot 0 + 1}{2} \int_{-1}^{1} f(x) P_0(x) = \frac{1}{2} \int_{-1}^{1} 2 \, dx = \frac{1}{2} \int_{-1}^{0} 2 + \frac{1}{2} \int_{-1}^{1} (-1) = \frac{1}{2}(2 - 1) = \frac{1}{2};$$

$$c_1 = \frac{2 \cdot 1 + 1}{2} \int_{-1}^{1} f(x) P_1(x) = \frac{3}{2} \int_{-1}^{1} f(x) \cdot x = \frac{3}{2} \int_{-1}^{0} 2x + \frac{3}{2} \int_{0}^{1} (-1)(x)$$

$$= \frac{3}{2} x^2 \bigg|_{-1}^{0} + \frac{3}{2} \left( -\frac{x^2}{2} \right) \bigg|_{0}^{1} = \frac{3}{2}(0^2 - (-1)^2) + \frac{3}{2} \cdot \frac{1 - 0}{2} = -\frac{3}{2} - \frac{3}{4} = -\frac{9}{4}.$$  

**Ans. to 1:** The first two coefficients are $c_0 = \frac{1}{2}$, $c_1 = -\frac{9}{4}$.

2. (5 points) Find product solutions, if possible, to the partial differential equation $\frac{\partial u}{\partial x} = 25 \frac{\partial u}{\partial y}$.

**Sol.** Try $u(x, y) = X(x)Y(y)$. So

$$X'(x)Y(y) = 25X(x)Y'(y).$$

Divided both sides by $X(x)Y(y)$:

$$\frac{X'(x)}{X(x)} = 25 \frac{Y'(y)}{Y(y)}.$$  

The left does not depend on $y$, and the right does not depend on $x$, and they are equal to each other, so neither depends on $x$ or $y$, so they are both equal to a number, let’s call it $k$. We have traded one pde with two odes:

$$\frac{X'(x)}{X(x)} = k$$

$$25 \frac{Y'(y)}{Y(y)} = k.$$  

These are

$$X'(x) - kX(x) = 0.$$
\[ Y'(y) - (k/25)Y(y) = 0. \]

The general solutions are
\[ X(x) = c_1 e^{kx}, \]
\[ Y(y) = c_2 e^{(k/25)y}. \]

So
\[ u(x, y) = (c_1 e^{kx})(c_2 e^{(k/25)y}) = (c_1 c_2)e^{kx}e^{(k/25)y}. \]

Renaming \( c_1 c_2, C, \) and simplifying we get
\[ u(x, y) = Ce^{kx+(k/25)y}. \]

\textbf{Ans. to 2:} \( u(x, y) = Ce^{kx+(k/25)y}, \) where \( C \) is an arbitrary constant.