Solutions to Dr. Z.’s Math 421(2), (Fall 2011, RU) REAL Quiz #4 (Oct. 20, 2011)

1. Find the Fourier series of \( f(x) = 2x^2 + 1 \) on the interval \((-\pi, \pi)\).

**Note:** You may use the ready-made indefinite integrals:

\[
\int x^2 \sin nx \, dx = \frac{(2 - n^2 x^2) \cos(nx) + 2nx \sin(nx)}{n^3}.
\]

\[
\int x^2 \cos nx \, dx = \frac{(n^2 x^2 - 2) \sin(nx) + 2xn \cos(nx)}{n^3}.
\]

**Sol.** The Fourier series of 1 is 1 and the Fourier series of \( x^2 \) is twice that of \( x^2 \), so let’s first find the Fourier series of \( x^2 \).

The Fourier Series, in general, is:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,
\]

where

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

Since \( x^2 \) is an **even** function, we can ignore \( b_n \) (they are automatically zero).

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{x^3}{3}\bigg|_{-\pi}^{\pi} = \frac{2\pi^2}{3}.
\]

Now, by the formula

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{(n^2 x^2 - 2) \sin(nx) + 2nx \cos(nx)}{n^3}\bigg|_{-\pi}^{\pi} = \frac{2\pi n(-1)^n}{n^3} = \frac{4(-1)^n}{n^2}.
\]

So the Fourier series of \( x^2 \) is

\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.
\]

**Finally** to get the Fourier series of \( 2x^2 + 1 \) we multiply by 2 and add 1:

\[
2x^2 + 1 = 1 + \frac{2\pi^2}{3} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.
\]
This is the ans.

Comments

1. An even more efficient way, since \( x^2 \) (and also \( 1 + 2x^2 \)) is an even function (note the pun), is to use
   \[
a_n = 2 \int_0^\pi f(x) \cos nx \, dx .
   \]

2. Many people gave the answer
   \[
   \frac{2\pi^3}{3} + 8 \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \cos nx + 1.
   \]
   They got full credit, but strictly speaking, the +1 should be together with the other number \( \frac{2\pi^3}{3} \). The pure number should stand in front.

3. Some people got the correct answer and then “improved” it by writing
   \[
   1 + \frac{2\pi^3}{3} + 8 \sum_{k=0}^\infty \frac{(-1)^n}{(2k+1)^2} \cos(2k+1)x
   \]
   This is wrong! (and they lost points for doing such nonsense. They blindly aped the simplification in one of the homework problem where there was \( 1 - (-1)^n \) at the top, but \( (-1)^n \) is never zero, so it is completely a different situation!

If in doubt, do not “simplify” (and take a chance of producing nonsense that will cost you important points (or your job, in real life).