1. Find the Fourier series of

\[ f(x) = \begin{cases} 
2, & \text{if } -\pi < x < 0; \\
-1, & \text{if } 0 \leq x < \pi.
\end{cases} \]

**Sol.** The Fourier Series is:

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \]

where

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \]

Let’s do the three integrations.

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx = 
\]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} 2 \, dx + \frac{1}{\pi} \int_{0}^{\pi} (-1) \, dx = \frac{1}{\pi} 2\pi + \frac{1}{\pi} (-\pi) = 2 - 1 = 1. \]

So \( a_0 = 1. \)

Next:

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx = 
\]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} 2 \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (-1) \cos nx \, dx = \frac{2 \sin nx}{n} \bigg|_{-\pi}^{0} - \frac{1 \sin nx}{n} \bigg|_{0}^{\pi} 
\]

\[ = 2 \sin 0 - \sin(-\pi n) - \sin 0 \frac{n}{\pi} + \sin \pi n \frac{n}{\pi} = 0. \]

So \( a_n = 0. \) Next:

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx = 
\]

\[ = \frac{1}{\pi} \int_{-\pi}^{0} 2 \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} (-1) \sin nx \, dx = \frac{-2 \cos 0 - \cos(-\pi n)}{n} + \frac{1}{\pi} \frac{\cos n\pi - \cos 0}{n} 
\]

\[ = \frac{-2 \cos 0 - \cos(-\pi n)}{n} + \frac{1}{\pi} \frac{\cos n\pi - \cos 0}{n} = \frac{-2 + 2(-1)^n}{n\pi} + \frac{(-1)^n - 1}{n\pi} = \frac{-2 + 2(-1)^n + (-1)^n - 1}{n\pi} = \frac{-3 + 3(-1)^n}{n\pi}. \]

Combining we get **First Ans.** The Fourier series of \( f(x) \) is

\[ f(x) = \frac{1}{2} - 3 \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin nx. \]
But when $n$ is even, $1 - (-1)^n = 0$ so let’s write $n = 2k + 1$ ($k = 0, 1, \ldots$), and note that when $n$ is odd $1 - (-1)^n = 2$ so we have \textbf{Better Ans.} The Fourier series of $f(x)$ is

$$f(x) = \frac{1}{2} - \frac{6}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \sin(2k+1)x.$$ 

\textbf{Comment:} People who gave the first (unsimplified) answer still got full credit. Unless it states “simplify as much as possible”, the former answer is acceptable.