1. Show that the given functions are orthogonal on the given interval.

\[ f_1(x) = e^{x^2} x^2 \quad , \quad f_2(x) = e^{-x^2} (x^3 + 2x) \quad , \quad [-1, 1] \]

Sol.

\[
(f_1, f_2) = \int_{-1}^{1} (e^{x^2} x^2) e^{-x^2} (x^3 + 2x) \, dx = \int_{-1}^{1} e^{x^2 - x^2} x^2 (x^3 + 2x) \, dx = \int_{-1}^{1} (x^5 + 2x^3) \, dx = 0 ,
\]

since the integrand \(x^5 + 2x^3\) is an odd function and the integration is symmetric. Since \((f_1, f_2) = 0\) it means that the two functions \(f_1(x), f_2(x)\) are orthogonal.

Comment: Almost everyone got it right!

2. Decide whether the set \(\{\cos x, \cos 2x, \cos 3x\}\) is orthogonal over the interval \([0, \pi/2]\).

Sol.

\[
(\cos x, \cos 2x) = \int_0^{\pi/2} \cos x \cos 2x \, dx .
\]

We have to use the famous trig. identity

\[ \cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B)) . \]

So

\[ \cos x \cos 2x = \frac{1}{2} (\cos 3x + \cos(-x)) = \frac{1}{2} (\cos 3x + \cos x) . \]

Integrating from 0 to \(\pi/2\):

\[
\int_0^{\pi/2} \cos x \cos 2x = \frac{1}{2} \int_0^{\pi/2} (\cos 3x + \cos x) = \frac{1}{2} \sin \frac{3x}{3} + \frac{1}{2} \sin x \\
= \left(\frac{1}{6} \sin 3x + \frac{1}{2} \sin x\right) \bigg|_0^{\pi/2} = \frac{1}{6} \sin \frac{3\pi}{2} - 0 + \frac{1}{2} \sin \frac{\pi}{2} - 0 = 0 .
\]

Remember that \(\sin \frac{\pi}{2} = 1\) and \(\sin \frac{3\pi}{2} = -1\), so this equals

\[ \frac{1}{6}(-1) + \frac{1}{2}(1) = \frac{1}{3} . \]

So the inner product \((\cos x, \cos 2x)\) is not zero, so these two functions are not orthogonal to each other, so there is no way that the whole family can be orthogonal, even if other pairs are OK.

Sol. to 2: The family is not orthogonal since we found two members, \(\cos x\), and \(\cos 2x\) that fail to be orthogonal to each other.
Comment 1: Quite a few people continued and found the inner products of the other two pairs, $(\cos x, \cos 3x)$ and $(\cos 2x, \cos 3x)$ and found out correctly, that these other inner-products are also not zero. This is a waste of time. To prove that a family is orthogonal, of course, you have to check every pair and make sure that the inner product is always zero, but in order to prove that a family of functions is not orthogonal, all you need is to come up with one pair whose inner-product is not zero.

To prove that a box of apples is a perfect, you need to check every apple. To prove that the box is not perfect, all you need is come up with one rotten apple.

Comment 2: About 40% of the people got it perfectly. Other people got it partially.