1. Show that the given functions are orthogonal on the given interval.

\[ f_1(x) = e^{x^2} x^7, \quad f_2(x) = e^{-x^2} (x^4 + x^2 + 1), \quad [-2, 2]. \]

**Sol.** We take the inner-product:

\[
\langle f_1(x), f_2(x) \rangle = \int_{-2}^{2} f_1(x) f_2(x) \, dx = \int_{-2}^{2} f_1(x) f_2(x) \, dx = \int_{-2}^{2} (e^{x^2} x^7)(e^{-x^2} (x^4 + x^2 + 1)) \, dx
\]

\[
= \int_{-2}^{2} e^{x^2 - x^2} x^7(x^4 + x^2 + 1) \, dx = \int_{-2}^{2} (x^{11} + x^9 + x^7) \, dx
\]

\[
= \frac{x^{12}}{12} + \frac{x^{10}}{10} + \frac{x^8}{8} \bigg|_{-2}^{2} = \frac{2^{12} - (-2)^{12}}{12} + \frac{2^{10} - (-2)^{10}}{10} + \frac{2^{8} - (-2)^{8}}{8} = 0.
\]

Since the inner product is zero, the two functions are orthogonal.

**Comment 1:** A faster way, once you get to \( \int_{-2}^{2} (x^{11} + x^9 + x^7) \, dx \) is to say that the integrand is an odd function and the interval of integration is symmetric (of the form \((-p, p)\), in this example, \(p = 2\)), so the integral must be zero.

**Comment 2:** Almost everyone got it right.

2. Show that \( \{\cos nx\}, \ n = 1, 2, \ldots \) is orthogonal over the interval \([0, \pi]\). Also find the norm of each function.

**Sol.** We need to take two different, typical members of this family, so let’s call them \(\cos nx\) and \(\cos mx\), where \(n \neq m\). We have to show that \(\langle \cos mx, \cos nx \rangle = 0\).

\[
\langle \cos mx, \cos nx \rangle = \int_{0}^{\pi} \cos mx \cos nx \, dx.
\]

We now use the famous trig, identity, recently entered into the cheatsheet:

\[
\cos A \cos B = \frac{1}{2} (\cos(A + B) + \cos(A - B)).
\]

So

\[
\cos mx \cos nx = \frac{1}{2} (\cos(m + n)x + \cos(m - n)x).
\]

Going back to the integral:

\[
\int_{0}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{0}^{\pi} (\cos(m + n)x + \cos(m - n)x) = \frac{1}{2} \left( \frac{\sin(m + n)x}{m + n} + \frac{\sin(m - n)x}{m - n} \right) \bigg|_{0}^{\pi} = \frac{1}{2} \left( \frac{\sin(m + n)\pi}{m + n} + \frac{\sin(m - n)\pi}{m - n} \right) = 0.
\]
\[
\frac{1}{2} \left( \frac{\sin(m+n)\pi}{m+n} + \frac{\sin(m-n)\pi}{m-n} \right) - \frac{1}{2} \left( \frac{\sin(m+n) \cdot 0}{m+n} + \frac{\sin(m-n) \cdot 0}{m-n} \right) = 0 - 0 = 0,
\]
Since both \( m+n \) and \( m-n \) are integers, so both \( \sin(m+n)\pi = 0 \) and \( \sin(m-n)\pi = 0 \). Of course \( \sin 0 = 0 \).

Since this is true for any two different members of the family, the family is indeed orthogonal.

It still remains to find the norm, \( ||\cos nx|| \). We first find the norm-squared.

\[
||\cos nx||^2 = \int_0^\pi \cos^2 nx \, dx.
\]
Using the famous trig-identity:

\[
\cos^2 A = \frac{1 + \cos 2A}{2},
\]
we get

\[
||\cos nx||^2 = \int_0^\pi \frac{1 + \cos 2nx}{2} \, dx = \frac{1}{2} \left( x + \frac{\sin 2nx}{2n} \right) \bigg|_0^\pi = \frac{1}{2} \left( \pi - 0 + \frac{\sin 2n(\pi) - \sin 0}{2n} \right) = \frac{\pi}{2}.
\]
Taking square-root:

\[
||\cos nx|| = \sqrt{\frac{\pi}{2}}.
\]

**Ans. to 2:** The family is indeed an orthogonal family of functions, and the norm of \( \cos nx \) is \( \sqrt{\frac{\pi}{2}} \).

**Comment:** About 50% of the people got it completely. Most people got the norm, but quite a few people messed up with the inner-product of \( \cos nx \) and \( \cos mx \).