1. Find the first three coefficients of the Fourier-Legendre expansion of

\[ f(x) = \begin{cases} 
2, & \text{if } -1 < x < 0; \\
3, & \text{if } 0 \leq x < 1. 
\end{cases} \]

**Sol.** Recall that

\[ P_0(x) = 1 \quad , \quad P_1(x) = 1 \quad , \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \]

\[ c_n = \frac{2n + 1}{2} \int_{-1}^{1} f(x)P_n(x) \, dx. \]

\[ c_0 = \frac{2 \cdot 0 + 1}{2} \int_{-1}^{1} f(x)P_0(x) \, dx = \frac{1}{2} \int_{-1}^{1} f(x) \, dx = \frac{1}{2} \int_{-1}^{0} f(x) \, dx + \frac{1}{2} \int_{0}^{1} f(x) \, dx. \]

\[ \frac{1}{2} \int_{-1}^{0} (2) \, dx + \frac{1}{2} \int_{0}^{1} (3) \, dx = \frac{1}{2} \cdot 2 \cdot dx + \frac{1}{2} \cdot 3 = \frac{5}{2}. \]

\[ c_1 = \frac{2 \cdot 1 + 1}{2} \int_{-1}^{1} f(x)P_1(x) \, dx = \frac{3}{2} \int_{-1}^{1} f(x) \, dx = \frac{3}{2} \int_{-1}^{0} f(x) \, dx + \frac{3}{2} \int_{0}^{1} f(x) \, dx. \]

\[ \frac{3}{2} \int_{-1}^{0} dx + \frac{3}{2} \int_{0}^{1} 3x \, dx = \frac{3}{2} \left( x^2 \right) \bigg|_{-1}^{0} + \frac{33}{2} \left( x^2 \right) \bigg|_{0}^{1} \]

\[ = \frac{3}{2} (0^2 - (-1)^2) + \frac{9}{4} (1^2 - 0^2) = \frac{3}{2} + \frac{9}{4} = \frac{3}{4}. \]

\[ c_2 = \frac{2 \cdot 2 + 1}{2} \int_{-1}^{1} f(x)P_2(x) \, dx = \frac{5}{2} \int_{-1}^{1} f(x) \left( \frac{3}{2}x^2 - \frac{1}{2} \right) \, dx = \frac{5}{2} \int_{-1}^{0} f(x) \left( \frac{3}{2}x^2 - \frac{1}{2} \right) \, dx + \frac{5}{2} \int_{0}^{1} f(x) \left( \frac{3}{2}x^2 - \frac{1}{2} \right) \, dx. \]

\[ \frac{5}{2} \int_{-1}^{0} 2 \left( \frac{3}{2}x^2 - \frac{1}{2} \right) \, dx + \frac{5}{2} \int_{0}^{1} 3 \left( \frac{3}{2}x^2 - \frac{1}{2} \right) \, dx \]

\[ = \frac{5}{2} \left( x^3 - x \right) \bigg|_{-1}^{0} + \frac{45}{4} \left( x^3 - \frac{1}{3} \right) \bigg|_{0}^{1} \]

\[ = \frac{5}{2} (-1 - (-1) - 0) + \frac{45}{4} \left( 1 - 0 \right) - \frac{15}{4} (1 - 0) = 0. \]

**Ans.** The first three coefficients of the Fourier-Legendre expansion of \( f(x) \) are \( c_0 = \frac{5}{2}, c_1 = \frac{3}{4}, c_2 = 0. \)

**Comment:** Only about %30 of the people got it completely correct, but many got \( c_0 \) and \( c_1 \) and most people did it the right way.

**Common mistake:** Many people wrote

\[ f(x) = \frac{5}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 \cdot P_2(x) \]
This is WRONG. If you want to write it in this format, you MUST write

\[ f(x) = \frac{5}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 \cdot P_2(x) + \ldots. \]

The \ldots are crucial! It means that the Fourier-Legendre series goes for ever, and what you got is the very start of an infinite journey.

Of course it is wrong that \( f(x) = \frac{5}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 \cdot P_2(x) \) (withut the \ldots). The right side is \( y = \frac{5}{2} + \frac{3}{4} x \) a straight line and not the discontinuous function of the problem.