Solutions to the Attendance Quiz for Nov. 3, 2011

1. Using the ready-made formula (don’t do it from scratch) solve the boundary value problem

\[ 4 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \quad t > 0, \]

\[ u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0 \]

\[ u(x, 0) = 3x(\pi - x), \quad 0 < x < \pi, \]

Hint (from Maple):

\[ \int_0^\pi x(\pi - x) \sin nx \, dx = \frac{2(1 - (-1)^n)}{n^3}. \]

Sol. The general formula, from the cheatsheet is (since \( k = 4 \) and \( L = \pi \)) is

\[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{-4n^2t} \sin nx, \]

where

\[ A_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx. \]

We have to compute \( A_n \).

\[ A_n = \frac{6}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx = \frac{6}{\pi} \cdot \frac{2(1 - (-1)^n)}{n^3} = \frac{12(1 - (-1)^n)}{\pi n^3}. \]

Putting it above:

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{12(1 - (-1)^n)}{\pi n^3} e^{-4n^2t} \sin nx. \]

This is a correct answer, but we can simplify it as follows. When \( n \) is even \((1 - (-1)^n)\) is always zero, when \( n \) is odd, it is always 2, so writing \( n = 2k + 1 \) \((k = 0, 1, 2, \ldots)\):

\[ u(x, t) = \sum_{k=0}^{\infty} \frac{24}{\pi(2k + 1)^3} e^{-4(2k+1)^2t} \sin(2k + 1)x. \]

and this is a better answer. Finally, we have the freedom to rename \( k \) \( n \), so yet-another way of writing the same answer is:

\[ u(x, t) = \sum_{n=0}^{\infty} \frac{24}{\pi(2n + 1)^3} e^{-4(2n+1)^2t} \sin(2n + 1)x. \]

Comment: About 30% got it completely. Many people forgot to take \( k = 4 \) and gave the answer as:

\[ u(x, t) = \sum_{n=0}^{\infty} \frac{24}{\pi(2n+1)^3} e^{-k(2n+1)^2t} \sin(2n + 1)x \]

(or the unsimplified version). Remember the general formula has \( k \), but each specific problem has its own (numerical) \( k \), and you have to figure it out.