1. Solve the heat equation $4u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0$ subject to

$$u(x,0) = \begin{cases} 1, & \text{if } |x| < 3; \\ 0, & \text{if } |x| > 3. \end{cases}$$

**Sol.** We first apply the Fourier transform $\mathcal{F}$, to the pde. Thinking of $x$ as the main variable, and $t$ as the parameter, and definining $U(\alpha, t) = \mathcal{F}\{u(x,t)\}$, we have

$$4\mathcal{F}\{u_{xx}\} = \mathcal{F}\{u_t\},$$

By the “dictionary”, we get the ode, in the variable $t$:

$$4(-\alpha^2)U(\alpha, t) = \frac{\partial U(\alpha, t)}{\partial t},$$

or in shorthand:

$$U'(t) + 4\alpha^2 U(t) = 0.$$

The general solution of this ode is

$$U(t) = c_1 e^{-4\alpha^2 t},$$

where $c_1$ is any constant. To find out what it is we need to know $U(0)$. But $U(0) = U(\alpha, 0)$ is the Fourier Transform of $u(x, 0)$, so we need to find

$$U(\alpha, 0) = \mathcal{F}\{u(x, 0)\} = \int_{-\infty}^{\infty} u(x,0)e^{i\alpha x} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \sin 3\frac{\alpha}{\alpha} e^{-4\alpha^2 t} e^{-i\alpha x} \, d\alpha = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin 3\alpha \, e^{-4\alpha^2 t} e^{-i\alpha x} \, d\alpha.$$

(Using the famous formula $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ that implies that $\frac{e^{iz} - e^{-iz}}{2i} = 2 \sin z$.)

So we have $U(0) = \frac{2\sin 3\alpha}{\alpha}$. Going back to the ode, we have $c_1 = \frac{2\sin 3\alpha}{\alpha}$, so

$$U(\alpha, t) = \frac{2\sin 3\alpha}{\alpha} e^{-4\alpha^2 t}.$$

Finally, to get the solution of our pde, $u(x,t)$, we apply $\mathcal{F}^{-1}$:

$$u(x,t) = \mathcal{F}^{-1}\left\{\frac{2\sin 3\alpha}{\alpha} e^{-4\alpha^2 t}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin 3\alpha e^{-4\alpha^2 t} e^{-i\alpha x} \, d\alpha = \frac{1}{\alpha} \int_{-\infty}^{\infty} \sin 3\alpha e^{-4\alpha^2} e^{-i\alpha x} \, d\alpha.$$

This is a correct answer, but it is slightly annoying, since it has imaginary numbers in it. Using Euler’s famous formula, $e^{iz} = \cos z + i \sin z$, we can write

$$u(x,t) = \frac{1}{\alpha} \int_{-\infty}^{\infty} \sin 3\alpha e^{-4\alpha^2 t} (\cos(\alpha x) - i \sin(\alpha x)) \, d\alpha.$$
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 3\alpha}{\alpha} e^{-4\alpha^2 t} \cos(\alpha x) \, d\alpha - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 3\alpha}{\alpha} e^{-4\alpha^2 t} \sin(\alpha x) \, d\alpha.
\]

But the second integral is an integral over \((-\infty, \infty)\) of an odd function, so it is automatically zero. So a **better looking** answer is:

\[
u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin 3\alpha}{\alpha} e^{-4\alpha^2 t} \cos(\alpha x) \, d\alpha.
\]

**Comment:** Do not try to “evaluate” this integral. It is (probably) impossible, and this is considered the final answer. It is giving the solution of the pde as an **integral representation**.