Dr. Z.'s Calc5 Cheatsheet

FINAL VERSION

[Note: This is the **only** sheet allowed in any of the quizzes and exams. No calculators of course!]

Version of Dec. 12 ,2011 (adding Euler's fromula $e^{iz} = \cos z + i \sin z$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$).

Previous Versions: Sept. 15, 2011: correcting errors in page 3, lines 9 and 10, last formula Oct. 12,2011: Adding two more useful trig. identities. Nov. 20 ,2011: (thanks to Eric Somers, correcting two typos in the trig identities). Nov. 22 ,2011 (thanks to Dr. Z, correcting typos in lines 3 and 4 of page 2)

Calc(-1) Reminders: The roots of $ax^2 + bx + c = 0$ are $(-b \pm \sqrt{b^2 - 4ac})/2a$.

Calc0 Reminders:

$$\sin^2 x + \cos^2 x = 1 \quad , \quad \sin(x+y) = \sin x \cos y + \cos x \sin y \quad , \quad \cos(x+y) = \cos x \cos y - \sin x \sin y \quad ,$$

$$\cos 2x = \cos^2 x - \sin^2 x \quad , \quad \sin 2x = 2 \sin x \cos x \quad , \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad , \quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad .$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B)) \quad , \quad \sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B)) \quad .$$

If n is an integer:

$$\sin n\pi = 0 \quad , \quad \cos n\pi = (-1)^n \quad , \quad \sin(n + \frac{1}{2})\pi = (-1)^n \quad , \quad \cos(n + \frac{1}{2})\pi = 0 \quad .$$

Calc1 Reminders: $(fg)' = f'g + fg'$, $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$, $(f(g(x)))' = f'(g(x))g'(x)$.
$(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\ln x)' = \frac{1}{x}$.

Calc2 Reminders: $\int uv' = uv - \int u'v$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, $\int e^{cx}\, dx = \frac{e^{cx}}{c} + C$ (if $c \neq 0$), $\int \sin(cx)\, dx = -\frac{cos(cx)}{c} + C$,$\int \cos(cx)\, dx = \frac{sin(cx)}{c} + C$, (if $c \neq 0$), $\int \frac{1}{x-a} dx = \ln|x-a| + C$.

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots \quad .$$

$$\cos x = 1 - \frac{x^2}{2} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \quad .$$

$$\sin x = x - \frac{x^3}{6} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \quad .$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2} x^2 + \ldots + \frac{a(a-1)\cdots(a-n+1)}{n!} x^n + \ldots \quad .$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots \quad .$$

$$\int xe^{cx}\,dx = \frac{(-1+cx)\,\mathrm{e}^{cx}}{c^2} + C \quad .$$

$$\int x^2e^{cx}\,dx = \frac{\left(2-2\,cx+x^2c^2\right)\mathrm{e}^{cx}}{c^3} + C \quad .$$

$$\int x^ne^{cx}\,dx = \frac{x^ne^{cx}}{c} - \frac{n}{c}\int x^{n-1}e^{cx}\,dx \quad .$$

$$\int_0^\infty x^ne^{-x}\,dx = n! \quad (n\ positive\ \ integer) \quad .$$

$$\int e^{bx}\sin(ax)\,dx = -a\frac{\cos(ax)\,\mathrm{e}^{bx}}{b^2+a^2} + b\frac{\sin(ax)\,\mathrm{e}^{bx}}{b^2+a^2} \quad .$$

$$\int e^{bx}\cos(ax)\,dx = \frac{b\cos(ax)\,\mathrm{e}^{bx}}{b^2+a^2} + a\frac{\sin(ax)\,\mathrm{e}^{bx}}{b^2+a^2} \quad .$$

$$\int x\cos(ax)\,dx = \frac{\cos(ax) + x\sin(ax)\,a}{a^2} \quad .$$

$$\int x\sin(ax)\,dx = \frac{\sin(ax) - x\cos(ax)\,a}{a^2} \quad .$$

$$\int x^n\sin(ax)\,dx = -\frac{x^n\cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx \quad .$$

$$\int x^n\cos(ax)\,dx = \frac{x^n\sin ax}{a} - \frac{n}{a}\int x^{n-1}\sin ax\,dx \quad .$$

$$\int_0^\infty x^ne^{-x}\,dx \ = \ n! \quad .$$

Polar $\rightarrow$ Rectangular: $x = r\cos\theta, y = r\sin\theta$; Rectangular $\rightarrow$ Polar: $r = \sqrt{x^2+y^2}, \theta = \tan^{-1}\frac{y}{x}$.

Calc3 Reminders: $(f_x)_y = (f_y)_x$, $grad\ f =< f_x, f_y, f_z >$, $div < F_1, F_2, F_3 >= (F_1)_x + (F_2)_y + (F_3)_z$, $curl < F_1, F_2, F_3 >=< (F_3)_y - (F_2)_z, (F_1)_z - (F_3)_x, (F_2)_x - (F_1)_y >$.

Calc4 Reminders:

The general solution of $ay''(x) + by'(x) + cy(x) = 0$ (a, b, c real numbers) is $y(x) = Ae^{\alpha x} + Be^{\beta x}$ if α, β are roots of $ar^2 + br + c = 0$ and they are real and distinct. If $\alpha = \beta$ then the general solution is $y(x) = Ae^{\alpha x} + Bxe^{\alpha x}$. If they are complex, $\mu \pm i\lambda$ then it is $y(x) = e^{\mu x}(A\cos\lambda x + B\sin\lambda x)$. In particular, the general solution of $y''(x) + \lambda^2 y(x) = 0$ is $y(x) = A\cos\lambda x + B\sin\lambda x$.

The general solution of $y''(x) - \lambda^2 y(x) = 0$ may be written either as $Ae^{\lambda x} + Be^{-\lambda x}$ or as $A\cosh\lambda x + B\sinh\lambda x$.

The **Cauchy-Euler** differential equation

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0 \quad ,$$

has the **general solution**

$$R(r) = C_1 r^n + C_2 r^{-n} \quad ,$$

when $n > 0$. When $n = 0$, the general solution is $R(r) = C_1 + C_2 \ln r$.

$e^{iz} = \cos z + i\sin z$, $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

Laplace Transform

$$F(s) = \int_0^\infty f(t)e^{-ts}\,dt \quad ,$$

$$\mathcal{L}\{1\} = \frac{1}{s} \quad , \quad \mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}} \quad (k = 1,2,3,...) \quad , \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad ,$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \quad , \quad \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \quad , \quad \mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \quad , \quad \mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2} \quad .$$

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 \quad , \quad \mathcal{L}^{-1}\{\frac{1}{s^k}\} = \frac{t^{k-1}}{(k-1)!} \quad (k = 1,2,3,...) \quad , \quad \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at} \quad ,$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2+k^2}\} = \frac{\sin kt}{k} \quad , \quad \mathcal{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos kt \quad , \quad \mathcal{L}^{-1}\{\frac{1}{s^2-k^2}\} = \frac{\sinh kt}{k} \quad , \quad \mathcal{L}^{-1}\{\frac{s}{s^2-k^2}\} = \cosh kt$$

$$\mathcal{L}\{y(t)\} = Y(s) \quad , \quad \mathcal{L}\{y'(t)\} = sY(s) - y(0) \quad , \quad \mathcal{L}\{y''(t)\} = s(sY(s) - y(0)) - y'(0) = s^2Y(s) - sy(0) - y'(0) \quad \ldots$$

$$\mathcal{L}\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0) \quad .$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \quad , \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

$$\mathcal{L}\{t^k e^{at}\} = \frac{k!}{(s-a)^{k+1}} \quad (k = 1,2,3,...) \quad , \quad \mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2} \quad , \quad \mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2} \quad .$$

$$\mathcal{L}^{-1}\{\frac{1}{(s-a)^k}\} = \frac{t^{k-1}e^{at}}{(k-1)!} \quad (k = 1,2,3,...) \quad , \quad \mathcal{L}^{-1}\{\frac{1}{(s-a)^2+k^2}\} = \frac{e^{at}\sin kt}{k} \quad , \quad \mathcal{L}^{-1}\{\frac{s-a}{(s-a)^2+k^2}\} = e^{at}\cos kt$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s) \quad (if\ a > 0) \quad .$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \quad (if\ a > 0) \quad .$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s) \quad (n\ pos.\ integer).$$

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \quad .$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s) \quad .$$

$$\mathcal{L}^{-1}(F(s)G(s) = (f * g)(t) \quad .$$

$$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = \frac{F(s)}{s} \quad ,$$

$$\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int_0^t f(\tau)\,d\tau$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0} \quad .$$

$$\mathcal{L}\{\delta(t)\} = 1 \quad .$$

Orthogonal Functions

Two functions $f(x)$ and $g(x)$ defined on an interval $[a,b]$ are **orthogonal** with respect to the **weight function** $w(x)$ if

$$\int_a^b f(x)g(x)\,w(x)dx = 0 \quad .$$

A set of functions $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ is an **orthogonal set** over $[a,b]$ with respect to the **weight** function $w(x)$ if the ϕ_i's are all orthogonal to each other, with respect to $w(x)$. In other words

$$\int_a^b \phi_m(x)\phi_n(x)\,w(x)dx = 0 \quad \textit{whenver} \quad m \neq n \quad .$$

The **inner-product** of two functions $(f(x), g(x))$ over $[a,b]$ with respect to the weight function $w(x)$ is

$$(f,g)_w = \int_a^b f(x)g(x)\,w(x)dx \quad .$$

The **norm-squared** of a function $f(x)$ on an interval $[a,b]$ with respect to the weight-function $w(x)$ is

$$||f||_w^2 = (f,f)_w = \int_a^b f(x)^2\,w(x)dx \quad .$$

A set of functions $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ is **orthonormal** over $[a,b]$ with respect to the weight-function $w(x)$ if it is orthogonal and the norms are all equal to 1.

Fourier Series (over $(-\pi,\pi)$)

If a function $f(x)$ is defined over the interval $(-\pi,\pi)$, then its **Fourier series** is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where the number a_0 is given

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad ,$$

and the numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad ,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad .$$

Fourier Series (over $(-L, L)$) find the function $g(x) = f(xL/\pi)$, that is defined over $(-\pi, \pi)$, and then go back using $f(x) = g(x\pi/L)$.

A function $f(x)$ is **even** if

$$f(-x) = f(x) \quad .$$

A function $f(x)$ is **odd** if

$$f(-x) = -f(x) \quad .$$

If $f(x)$ is even then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

If $f(x)$ is odd then $\int_{-a}^{a} f(x)\,dx = 0$.

Fourier Cosine Series (for Even Functions)

The Fourier series of an **even** function $f(x)$ on the interval $(-\pi, \pi)$ is the **cosine series** (no sines show up!)

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ,$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad .$$

Fourier Sine Series (for Odd Functions) The Fourier series of an **odd** function $f(x)$ on the interval $(-\pi, \pi)$ is the **sine series** (no cosines show up!)

$$\sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad .$$

Half Range Expansion If a function $f(x)$ is only defined on $(0, \pi)$, then we can extend it to $(-\pi, \pi)$ to either get an even function, and find its **cosine series**, or to an odd function and get its **sine series**. Both of them are supposed to converge to $f(x)$ in $(0, \pi)$.

The **complex Fourier series** of a function f defined on the interval $(-\pi, \pi)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad ,$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

The **complex Fourier series** of a function f defined on a general interval $(-p, p)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p} \quad ,$$

where

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in\pi x/p}\, dx \quad , \quad n = 0, \pm 1, \pm 2, \ldots \quad .$$

Sturm-Liouville Problem

A **Regular Sturm-Liouville Problem** on an interval $[a, b]$ is a **differential equation** of the form

$$\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \quad ,$$

subject to the **boundary conditions**

$$A_1 y(a) + B_1 y'(a) = 0 \quad ,$$

$$A_2 y(b) + B_2 y'(b) = 0 \quad .$$

Here p, q, r are continuous functions, and in addition $r'(x)$ should also be continuous. Also we need $r(x) > 0$ and $p(x) > 0$ on the interval $[a, b]$.

Singular Sturm-Liouville Problem on an interval $[a, b]$ is a **differential equation** of the above form but the condition that $r(x) > 0$ in $[a, b]$ is not always true, but then you only use some of the boundary conditions.

For most λ's there is **no solution** (except for the "trivial solution" $y(x) = 0$). Those lucky ones for which there is a non-zero solution are called **eigenvalues** and the corresponding solutions are called **eigenfunctions**.

Sturm-Liouville Theorem: 1. For a regular Sturm-Liouville problem there exist an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

such that $\lambda_n \to \infty$.

2. Each **eigenvalue** λ_i has just one corresponding eigenfunction $y_i(x)$ (up to a constant multiple)

3. All the eigenfunctions are **linearly independent**. In other words, there is no way that you can express one of them as a linear combination of other ones.

4. The eigenfunctions $\{y_i(x)\}$ are **orthogonal** over $[a, b]$ with respect to the **weight-function** $p(x)$.

Fourier-Legendre Series

The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are defined by the **generating function**

$$\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2} \quad .$$

Another way to define them is via the **recurrence**

$$P_n(x) = \frac{2n-1}{n} x P_{n-1}(x) - \frac{n-1}{n} P_{n-2}(x) \quad ,$$

subject to the **initial values**:

$$P_0(x) = 1 \quad P_1(x) = x \quad .$$

The **Fourier-Legendre series** of a function $f(x)$ defined on the interval $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \quad ,$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \quad .$$

Heat Equation

1. Both ends are at temperature 0:

The solution of

$$k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0$$

subject to

$$u(0, t) = 0 \quad , \quad u(L, t) = 0 \quad , \quad t > 0$$

$$u(x, 0) = f(x) \quad , \quad 0 < x < L \quad ,$$

is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x \quad ,$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x\, dx \quad .$$

2. Both ends are insulated

The solution of

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad , \quad 0 < x < L \quad , \quad t > 0$$

subject to

$$u_x(0,t) = 0 \quad , \quad u_x(L,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = f(x) \quad , \quad 0 < x < L \quad ,$$

is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \cos\frac{n\pi}{L}x \quad ,$$

where

$$A_0 = \frac{2}{L}\int_0^L f(x)\,dx \quad , \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx \quad .$$

Wave Equation (Special case: $L = \pi$)

The solution of the boundary value wave equation

$$a^2 u_{xx} = u_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad 0 < x < \pi \quad .$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(nat) + B_n \sin(nat)\right)\sin(nx) \quad ,$$

where the numbers A_n and B_n are given by the formulas

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \quad ,$$

$$B_n = \frac{2}{n\pi a}\int_0^\pi g(x)\sin nx\,dx.$$

Wave Equation (General Case)

The solution of the boundary value wave equation

$$a^2 u_{xx} = u_{tt} \quad , \quad 0 < x < L \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad 0 < x < L \quad .$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\frac{n\pi a}{L} t) + B_n \sin(\frac{n\pi a}{L} t) \right) \sin(\frac{n\pi}{L} x) \quad ,$$

where the numbers A_n and B_n are given by the formulas

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx \quad ,$$

$$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

Boundary Superposition Principle for the 2D Laplace's Equation

If you have a complicated so-called *Dirichlet* boundary value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad ,$$

$$u(0,y) = F(y) \quad , \quad u(a,y) = G(y) \quad , \quad 0 < y < b \quad .$$

$$u(x,0) = f(x) \quad , \quad u(x,b) = g(x) \quad , \quad 0 < x < a \quad .$$

You **break-it up** into two problems as follows.

First Problem: Find the solution, let's call it $u_1(x,y)$ satisfying

$$(u_1)_{xx} + (u_1)_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad ,$$

$$u_1(0,y) = 0 \quad , \quad u_1(a,y) = 0 \quad , \quad 0 < y < b \quad ,$$

$$u_1(x,0) = f(x) \quad , \quad u_1(x,b) = g(x) \quad , \quad 0 < x < a \quad .$$

Second Problem: Find the solution, let's call it $u_2(x,y)$ satisfying

$$(u_2)_{xx} + (u_2)_{yy} = 0 \quad , \quad 0 < x < a \quad , \quad 0 < y < b \quad ,$$

$$u_2(0,y) = F(y) \quad , \quad u_2(a,y) = G(y) \quad , \quad 0 < y < b \quad ,$$

$$u_2(x,0) = 0 \quad , \quad u_2(x,b) = 0 \quad , \quad 0 < x < a \quad .$$

Once you solved these (already complicated!) two problems, the **final** solution, to the original problem, is simply

$$u(x,y) = u_1(x,y) + u_2(x,y) \quad .$$

In other words, just add them up!

Laplace's Equation in Polar Coordinates

The Laplacian Equation in two dimensions

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u(x,y)=0 \quad ,$$

phrased in the usual **rectangular coordinates** (x,y), becomes, in **polar coordinates** (r,θ),

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)u(r,\theta)=0 \quad .$$

Laplace Transform for 2D PDEs:

If $\mathcal{L}\{u(x,t)\}=U(x,s)$, then

$$\mathcal{L}\{\frac{\partial u}{\partial t}\} \;=\; sU(x,s)-u(x,0) \quad ,$$

$$\mathcal{L}\{\frac{\partial^2 u}{\partial t^2}\} \;=\; s^2U(x,s)-su(x,0)-u_t(x,0) \quad .$$

$$\mathcal{L}\{\frac{\partial u}{\partial x}\} \;=\; \frac{\partial U(x,s)}{\partial x}$$

$$\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\} \;=\; \frac{\partial^2 U(x,s)}{\partial x^2}$$

Fourier Integral

The **Fourier Integral** of a function $f(x)$ defined on the real line $(-\infty,\infty)$ is given by

$$\frac{1}{\pi}\int_0^\infty [\,A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x\,]\,d\alpha \quad ,$$

where

$$A(\alpha)=\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$$

$$B(\alpha)=\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

Fourier Transform:

$$\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Transform:

$$\mathcal{F}^{-1}\{F(\alpha)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}\,d\alpha \;=\; f(x) \quad .$$

Fourier Sine Transform:

$$\mathcal{F}_s\{f(x)\} = \int_0^\infty f(x)\sin\alpha x\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Sine Transform:

$$\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha \;=\; f(x) \quad .$$

Fourier Cosine Transform:

$$\mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\cos\alpha x\,dx \;=\; F(\alpha) \quad .$$

Inverse Fourier Cosine Transform:

$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos\alpha x\,d\alpha \;=\; f(x) \quad .$$

If $\mathcal{F}\{f(x)\} = F(\alpha)$ then for $n = 1, 2, 3, \ldots$.

$$\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha) \quad .$$

If $\mathcal{F}_s\{f(x)\} = F(\alpha)$ then

$$\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0) \quad .$$

If $\mathcal{F}_c\{f(x)\} = F(\alpha)$ then

$$\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0) \quad .$$

Euler's method for Numerically solving a first-order ode

For the **initial value problem**

$$y' = f(x, y) \quad , \quad y(x_0) = y_0 \quad ,$$

with **mesh-size** h, you define

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \ldots \quad ,$$

and compute, **one-step-at-a-time**

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad , \quad n = 1, 2 \ldots .$$

y_n is an **approximation** for $y(x_n)$. The smaller h, the better the approximation.

The Improved Euler method for Numerically solving a first-order ode

To solve the **initial value problem**

$$y' = f(x,y) \quad , \quad y(x_0) = y_0$$

with **mesh-size** h, you define

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \dots \quad ,$$

and compute, **one-step-at-a-time**

$$y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad ,$$

$$y_n = y_{n-1} + h\frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2} \quad , \quad n = 1, 2 \dots .$$

Then y_n is an **approximation** for $y(x_n)$. The smaller h, the better the approximation.

Fourth-Order Runge-Kutta (RK4)

To approximate solutions of

$$y' = f(x,y) \quad , \quad y(x_0) = y_0 \quad ,$$

at $x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$ do the following, starting at y_0, for $n = 1, 2, \dots$

$$k_1 = f(x_{n-1}, y_{n-1})$$
$$k_2 = f(x_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hk_1)$$
$$k_3 = f(x_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hk_2)$$
$$k_4 = f(x_{n-1} + h, y_{n-1} + hk_3) \quad ,$$

and finally

$$y_n = y_{n-1} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad .$$

Discretization of PDEs

The **discrete approximations** of the second derivatives with **mesh-size** h are:

$$u_{xx} \approx \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] \quad ,$$

$$u_{yy} \approx \frac{1}{h^2}[u(x,y+h) - 2u(x,y) + u(x,y-h)] \quad .$$

Numerical Solution of 2D Laplacian Dirichlet problems

The **five-point approximation** of the Laplacian $u_{xx} + u_{yy}$ (in 2D) is

$$u_{xx} + u_{yy} \approx \frac{1}{h^2}[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)]$$

To **numerically** (approximately) solve the **Dirichlet** problem $u_{xx} + u_{yy} = 0$ in a region D with **boundary condition** $u(x, y) = F(x, y)$ along the boundary with mesh-size h, you set $u_{i,j} = u(ih, jh)$ and set-up a system of linear equation as follows.

For each (ih, jh) **inside** the region, you have an equation

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0 \quad ,$$

and for every **boundary point**

$$u_{i,j} = F(ih, jh) \quad .$$

Then do the linear algebra, and the solutions, $\{u_{i,j}\}$ would give you approximations for the values of the "real thing" at the interior points $\{(ih, jh)\}$.