Important Definition: If a function \( f(x) \) is defined over the interval \((-\pi, \pi)\), then its Fourier series is

\[
a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx ,
\]

where the number \( a_0 \) is given

\[
a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx ,
\]

and the numbers \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) are given by:

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

Important Theorem: If \( f(x) \) and \( f'(x) \) are continuous on \((-\pi, \pi)\) then the Fourier series of \( f(x) \) converges to it.

Note: If \( f(x), f'(x) \) are only piece-wise continous then the Fourier series converges to \( f(x) \) at all the good points, and at the “breaking-points” it converges to the average of the limit from the left and the limit from the right.

Note: If the function \( f(x) \) is defined over an interval \((-p, p)\) that is not \((-\pi, \pi)\), you can still define a Fourier series, but first you consider \( g(x) = f(xp/\pi) \), find the Fourier series for \( g(x) \) and then go back to \( f(x) \) by using \( f(x) = g(xp/\pi) \). At the end you would get an expansion in \( \sin(\frac{nx}{p}x) \) and \( \cos(\frac{nx}{p}x) \).

Problem 8.1: Find the Fourier series of

\[
f(x) = \begin{cases} 
1, & \text{if } -\pi < x < 0; \\
-3, & \text{if } 0 \leq x < \pi .
\end{cases}
\]

Solution:

\[
a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{0}^{0} f(x) \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \, dx = \\
\frac{1}{\pi} \left[ (1) \right]_{x=0}^{x=\pi} - \frac{3}{\pi} \left[ (1) \right]_{x=0}^{x=\pi} = \frac{1}{\pi} ((0 - (-\pi)) - 3(\pi - 0)) = -2 .
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_{0}^{0} f(x) \cos nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx = \\
\frac{1}{\pi} \left[ \cos nx \right]_{x=0}^{x=\pi} - \frac{3}{\pi} \left[ \cos nx \right]_{x=0}^{x=\pi} = \frac{1}{\pi} \left( \sin nx \right)_{x=0}^{x=\pi} - \frac{3}{\pi} \left( \sin nx \right)_{x=0}^{x=\pi} =
\]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx - 3 \frac{1}{\pi} \int_{0}^{\pi} \sin nx \, dx = \left( -\frac{1}{n\pi} \cos nx \right) \bigg|_{0}^{\pi} + \frac{3}{n\pi} \left( \cos nx \right) \bigg|_{0}^{\pi} \]
\[ = -\frac{1}{n\pi} (\cos(0) - \cos(-n\pi)) + \frac{3}{n\pi} (\cos(n\pi) - \cos(0)) = \frac{-1}{n\pi} (1 - (-1)^n) + \frac{3}{n\pi} ((-1)^n - 1) = \frac{4}{n\pi} ((-1)^n - 1). \]

(Recall that \( \cos n\pi = (-1)^n \) for all integers \( n \).) Combining we get that the Fourier series of \( f(x) \) is:

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} 2x \, dx = 2 \frac{x^2}{2} \bigg|_{-\pi}^{\pi} = 0 \]
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx = \frac{1}{\pi} \left( \frac{2}{\pi} \right) \int_{-\pi}^{\pi} x \cos nx \, dx = 0 \]

(since \( x \cos nx \) is an odd function and the integration is symmetric (over \( (-a, a) \) for some \( a \) in this case \( a = \pi \)). Of course you can do it by parts the long way.)

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx = \frac{1}{\pi} \left( \frac{2}{\pi} \right) \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi^2} \int_{-\pi}^{\pi} x \sin nx \, dx \]

Remember how to integrate

\[ \int x \sin nx \, dx \]
by integration by parts. \( u = x, \; v' = \sin nx \). So \( u' = 1 \) and \( v = -\frac{\cos nx}{n} \) and we get

\[ \int x \sin nx \, dx = x \left( -\frac{\cos nx}{n} \right) - \int u' v \, dx = x \left( -\frac{\cos nx}{n} \right) - \int (1) \left( -\frac{\cos nx}{n} \right) \, dx \]
\[ = \frac{1}{n} (\cos nx) + \int \frac{\cos nx}{n} \, dx = \frac{1}{n} (\cos nx) + \frac{\sin nx}{n^2} \]
So
\[\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{-1}{n} (\pi \cos n\pi - (-\pi) \cos (-n\pi)) + (0 - 0) = \frac{-1}{n} (2\pi \cos n\pi) = \frac{-2\pi}{n} (-1)^n.\]

Going back to \(b_n\), we have
\[b_n = \left(\frac{2}{\pi^2}\right) \frac{-2\pi}{n} (-1)^n = \frac{4(-1)^{n+1}}{\pi n}.\]

So the Fourier series of \(g(x)\) is:
\[a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = 0 + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin nx = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin nx\]

Going back to \(f(x)\), since \(f(x) = g(x\frac{\pi}{2})\), all we do is replace \(x\) by \(x\frac{\pi}{2}\)
\[\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin\left(\frac{n\pi}{2}x\right).\]

**Ans. to 8.2:** The Fourier series of \(f(x) = x\) on the interval \((-2, 2)\) is \(\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin\left(\frac{n\pi}{2}x\right).\)