**Dr. Z.'s Calc5 Lecture 8 Handout: Fourier Series**

By Doron Zeilberger

**Important Defintion**: If a function $f(x)$ is defined over the interval $(-\pi,\pi)$, then its **Fourier series** is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

where the number $a_0$ is given

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad ,$$

and the numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

**Important Theorem:** If $f(x)$ and $f'(x)$ are continuous on $(-\pi,\pi)$ then the Fourier series of $f(x)$ converges to it.

**Note:** If $f(x), f'(x)$ are only piece-wise continous then the Fourier series converges to $f(x)$ at all the good points, and at the "breaking-points" it converges to the average of the limit from the left and the limit from the right.

**Note:** If the function $f(x)$ is defined over an interval $(-p,p)$ that is not $(-\pi,\pi)$, you can still define a Fourier series, but first you cosider $g(x)=f(xp/\pi)$, find the Fourier series for $g(x)$ and then go back to $f(x)$ by using $f(x)=g(x\frac{\pi}{p})$. At the end you would get an expansion in $\sin(\frac{n\pi}{p}x)$ and $\cos(\frac{n\pi}{p}x)$.

**Problem 8.1**: Find the Fourier series of

$$f(x)=\begin{cases} 1, & \text{if } -\pi < x < 0; \\ -3, & \text{if } 0 \le x < \pi. \end{cases}$$

**Solution**:

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx =$$

$$\frac{1}{\pi}\int_{-\pi}^{0}(1)\,dx - \frac{3}{\pi}\int_{0}^{\pi}(1)\,dx = \frac{1}{\pi}((0-(-\pi))-3(\pi-0)) = -2 \quad .$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx - \frac{3}{\pi}\int_{0}^{\pi}\cos nx\,dx = \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)\Big|_{-\pi}^{0} - \frac{3}{\pi}\left(\frac{\sin nx}{n}\right)\Big|_{0}^{\pi}$$

$$= \frac{1}{n\pi}(\sin(0) - \sin(-n\pi)) - \frac{3}{n\pi}(\sin(n\pi) - \sin(0)) = 0$$

(Recall that $\sin n\pi = 0$ for all integers $n$.)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\sin nx\,dx - \frac{3}{\pi}\int_{0}^{\pi}\sin nx\,dx = \frac{-1}{\pi}\left(\frac{\cos nx}{n}\right)\Big|_{-\pi}^{0} + \frac{3}{\pi}\left(\frac{\cos nx}{n}\right)\Big|_{0}^{\pi}$$

$$= \frac{-1}{n\pi}(\cos(0) - \cos(-n\pi)) + \frac{3}{n\pi}(\cos(n\pi) - \cos(0)) = \frac{-1}{n\pi}(1-(-1)^n) + \frac{3}{n\pi}((-1)^n - 1) = \frac{4}{\pi n}((-1)^n - 1) \quad .$$

(Recall that $\cos n\pi = (-1)^n$ for all integers $n$.) Combining we get that the Fourier series of $f(x)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ,$$

$$= \frac{-2}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{\pi n}\sin nx = -1 + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{\pi n}\sin nx \quad .$$

**Ans. to 8.1**: The Fourier series is: $-1 + \sum_{n=1}^{\infty} \frac{4((-1)^n - 1)}{\pi n}\sin nx$.

**Problem 8.2**: Find the Fourier series of $f(x) = x$ on the interval $(-2, 2)$.

**Solution**. We consider $g(x) = f(\frac{2x}{\pi}) = \frac{2}{\pi}x$ that is defined on $(-\pi, \pi)$.

$$a_0 := \frac{1}{\pi}\frac{2}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{2}{\pi^2}\frac{x^2}{2}\Big|_{-\pi}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = (\frac{1}{\pi})(\frac{2}{\pi})\int_{-\pi}^{\pi} x\cos nx\,dx = 0$$

(since $x \cos nx$ is an **odd** function and the integration is symmetric (over $(-a, a)$ for some $a$ in this case $a = \pi$). Of course you can do it by parts the long way.)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx = (\frac{1}{\pi})(\frac{2}{\pi})\int_{-\pi}^{\pi} x\sin nx\,dx = (\frac{2}{\pi^2})\int_{-\pi}^{\pi} x\sin nx\,dx$$

Remember how to integtate

$$\int x\sin nx\,dx$$

by integration by parts. $u = x$, $v' = \sin nx$. So $u' = 1$ and $v = \frac{-\cos nx}{n}$ and we get

$$\int x\sin nx\,dx = x\frac{-\cos nx}{n} - \int u'v\,dx = x\frac{-\cos nx}{n} - \int (1)\frac{-\cos nx}{n}\,dx$$

$$= \frac{1}{n}(-x\cos nx) + \int \frac{\cos nx}{n}\,dx = \frac{-1}{n}(x\cos nx) + \frac{\sin nx}{n^2}$$

So

$$\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{-1}{n}(\pi \cos n\pi - (-\pi)\cos(-n\pi)) + (0-0) = \frac{-1}{n}(2\pi \cos n\pi) = \frac{-2\pi}{n}(-1)^n \quad .$$

Going back to $b_n$, we have

$$b_n = (\frac{2}{\pi^2})\frac{-2\pi}{n}(-1)^n = \frac{4(-1)^{n+1}}{\pi n} \quad .$$

So the Fourier series of $g(x)$ is:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \frac{0}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin nx \quad = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin nx$$

Going back to $f(x)$, since $f(x) = g(x\frac{\pi}{2})$, all we do is replace $x$ by $x\frac{\pi}{2}$

$$\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin(\frac{n\pi}{2}x) \quad .$$

**Ans. to 8.2**: The Fourier series of $f(x) = x$ on the interval $(-2, 2)$ is $\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi n} \sin(\frac{n\pi}{2}x)$.