Dr. Z.'s Calc5 Lecture 6 Handout:

Using the Laplace Transform to solve Systems of Linear Differential Equations

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Systems of ODEs

When we have one differential equation, with one unknown function $y(t)$ and we use the Laplace Transform method, we trade it with one algebraic equation in $Y(s)$. When we have several equations with several unknown functions we trade it with a system of algebraic equations in the Laplace Transforms. We then solve the system, and at the end, transform back.

Problem 6.1: Solve the system

$$\frac{dx}{dt} = -x + y,$$
$$\frac{dy}{dt} = 2x,$$
$$x(0) = 0, \quad y(0) = 1.$$

Solution: Let $\mathcal{L}\{x\} = X(s), \mathcal{L}\{y\} = Y(s)$. Applying $\mathcal{L}$ to both equations we get

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\},$$
$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = 2\mathcal{L}\{x\}.$$

Now $\mathcal{L}\left\{\frac{dx}{dt}\right\} = sX(s) - x(0) = sX(s)$, $\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 1 = sY(s) - 1$

So we have the system

$$sX = -X + Y,$$
$$sY - 1 = 2X.$$

Cleaning up:

$$(s + 1)X - Y = 0,$$
$$-2X + sY = 1.$$

From the first equation we get

$$Y = (s + 1)X.$$

Substituting this into the second equation:

$$-2X + s(s + 1)X = 1$$

Factoring out $X$:

$$(-2 + s(s + 1))X = 1$$

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Solving for $X$, we get
\[ X = \frac{1}{s^2 + s - 2} = \frac{1}{(s + 2)(s - 1)} . \]
Since $Y = (s + 1)X$, we have
\[ Y = \frac{s + 1}{(s + 2)(s - 1)} . \]
So we found
\[ X(s) = \frac{1}{(s + 2)(s - 1)} , \quad Y(s) = \frac{s + 1}{(s + 2)(s - 1)} . \]

Finally, we need to find $x(t)$ and $y(t)$.
\[ x(t) = \mathcal{L}^{-1}\left\{ \frac{1}{(s + 2)(s - 1)} \right\} , \quad y(t) = \mathcal{L}^{-1}\left\{ \frac{s + 1}{(s + 2)(s - 1)} \right\} \]

By partial fractions:
\[ \frac{1}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1} , \quad \frac{1}{(s + 2)(s - 1)} = \frac{A(s - 1) + B(s + 2)}{(s + 2)(s - 1)} . \]
So $1 = A(s - 1) + B(s + 2)$. When $s = 1$, $1 = B \cdot (1 + 2)$ so $B = \frac{1}{3}$. When $s = -2$, $1 = A(-2 - 1)$ so $A = -\frac{1}{3}$. So
\[ X = -\frac{1}{3} \cdot \frac{1}{s + 2} + \frac{1}{3} \cdot \frac{1}{s - 1} . \]
So
\[ x(t) = \mathcal{L}^{-1}\{ -\frac{1}{3} \cdot \frac{1}{s + 2} + \frac{1}{3} \cdot \frac{1}{s - 1} \} = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t . \]

As for $Y(s)$:
\[ \frac{s + 1}{(s + 2)(s - 1)} = \frac{A}{s + 2} + \frac{B}{s - 1} , \quad \frac{s + 1}{(s + 2)(s - 1)} = \frac{A(s - 1) + B(s + 2)}{(s + 2)(s - 1)} . \]
So $s + 1 = A(s - 1) + B(s + 2)$. When $s = 1$, $1 + 1 = B \cdot (1 + 2)$ so $B = \frac{2}{3}$. When $s = -2$, $-2 + 1 = A(-2 - 1)$ so $A = \frac{1}{3}$. So
\[ Y = \frac{1}{3} \cdot \frac{1}{s + 2} + \frac{2}{3} \cdot \frac{1}{s - 1} . \]
So
\[ y(t) = \mathcal{L}^{-1}\{ \frac{1}{3} \cdot \frac{1}{s + 2} + \frac{2}{3} \cdot \frac{1}{s - 1} \} = \frac{1}{3} e^{-2t} + \frac{2}{3} e^t . \]

Ans. to 6.1: $x(t) = -\frac{1}{3} e^{-2t} + \frac{1}{3} e^t$ , $y(t) = \frac{1}{3} e^{-2t} + \frac{2}{3} e^t$ .

Problem 6.2: Solve the system
\[ \frac{d^2 x}{dt^2} = \frac{1}{2}(-x + y) , \]
\[ \frac{d^2 y}{dt^2} = \frac{1}{2}(x - y) , \]
Solution: Let $L\{x\} = X(s), L\{y\} = Y(s)$. Applying $L$ to both equations we get

\[
L\left(\frac{d^2x}{dt^2}\right) = -\frac{1}{2}L\{x\} + \frac{1}{2}L\{y\},
\]

\[
L\left(\frac{d^2y}{dt^2}\right) = \frac{1}{2}L\{x\} - \frac{1}{2}L\{y\},
\]

Now $L\left(\frac{d^2x}{dt^2}\right) = s^2X(s) - x(0)s - x'(0) = s^2X(s) - 2, L\left(\frac{d^2y}{dt^2}\right) = s^2Y(s) - y(0)s - y'(0) = s^2Y(s)$

So we have the system

\[
s^2X - 2 = -\frac{1}{2}X + \frac{1}{2}Y
\]

\[
s^2Y = \frac{1}{2}X - \frac{1}{2}Y
\]

Cleaning up:

\[
(s^2 + \frac{1}{2})X - \frac{1}{2}Y = 2,
\]

\[
(s^2 + \frac{1}{2})Y = \frac{1}{2}X.
\]

From the second equation we get

\[
X = (2s^2 + 1)Y.
\]

Substituting this into the first equation:

\[
(s^2 + \frac{1}{2})(2s^2 + 1)Y - \frac{1}{2}Y = 2,
\]

Factor out $Y$:

\[
(s^2 + \frac{1}{2})(2s^2 + 1)Y - \frac{1}{2}Y = 2,
\]

\[
(2s^4 + 2s^2 + \frac{1}{2} - \frac{1}{2})Y = 2,
\]

\[
(2s^4 + 2s^2)Y = 2,
\]

\[
2s^2(s^2 + 1)Y = 2.
\]

So

\[
Y = \frac{1}{s^4(s^2 + 1)}
\]

Going back to $X$:

\[
X = \frac{2s^2 + 1}{s^4(s^2 + 1)}
\]
Finally, we need to find $x(t)$ and $y(t)$.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s^2 + 1}{s^2(s^2 + 1)}\right\}, \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\}$$

By partial fractions:

$$X = \frac{2s^2 + 1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2 + 1}$$

$$2s^2 + 1 = (As + B)(s^2 + 1) + Cs^2 = As^3 + Bs^2 + As + B + C s^2 = As^3 + (B + C)s^2 + As + B.$$  

Comparing coefficients, we get:

$$A = 0, \quad B + C = 2, \quad A = 0, \quad B = 1,$$

so $A = 0$, $B = 1$, $C = 1$ and we have

$$X = \frac{1}{s^2} + \frac{1}{s^2 + 1}$$

So

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{s^2 + 1}\right\} = t + \sin t$$

As for $Y(s)$:

$$Y = \frac{1}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2 + 1}$$

$$1 = (As + B)(s^2 + 1) + Cs^2 = As^3 + Bs^2 + As + B + C s^2 = As^3 + (B + C)s^2 + As + B.$$  

Comparing coefficients, we get:

$$A = 0, \quad B + C = 0, \quad A = 0, \quad B = 1,$$

so $A = 0$, $B = 1$, $C = -1$ and we have

$$Y = \frac{1}{s^2} - \frac{1}{s^2 + 1}.$$  

So

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2 + 1}\right\} = t - \sin t.$$  

**Ans. to 6.2**: $x(t) = t + \sin t$, $y(t) = t - \sin t$. 

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