Dr. Z.'s Calc5 Lecture 6 Handout:
Using the Laplace Transform to solve Systems of Linear Differential Equations

By Doron Zeilberger

Systems of ODEs

When we have **one** *differential* equation, with one unknown function $y(t)$ and we use the Laplace Transform method, we trade it with **one** *algebraic* equation in $Y(s)$. When we have **several** equations with several unknown functions we trade it with a system of algebraic equations in the Laplace Transforms. We then solve the system, and at the end, transform back.

Problem 6.1: Solve the system

$$\frac{dx}{dt} = -x + y \quad ,$$

$$\frac{dy}{dt} = 2x \quad ,$$

$$x(0) = 0 \quad , \quad y(0) = 1 \quad .$$

Solution: Let $\mathcal{L}\{x\} = X(s), \mathcal{L}\{y\} = Y(s)$. Applying $\mathcal{L}$ to both equations we get

$$\mathcal{L}\{\frac{dx}{dt}\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\} \quad ,$$

$$\mathcal{L}\{\frac{dy}{dt}\} = 2\mathcal{L}\{x\} \quad .$$

Now $\mathcal{L}\{\frac{dx}{dt}\} = sX(s) - x(0) = sX(s)$, $\mathcal{L}\{\frac{dy}{dt}\} = sY(s) - y(0) = sY(s) - 1 = sY(s) - 1$

So we have the system

$$sX = -X + Y$$

$$sY - 1 = 2X$$

Cleaning up:

$$(s+1)X - Y = 0 \quad ,$$

$$-2X + sY = 1 \quad .$$

From the first equation we get

$$Y = (s+1)X \quad .$$

Substituting this into the second equation:

$$-2X + s(s+1)X = 1$$

Factoring out X:

$$(-2 + s(s+1))X = 1$$

Solving for X, we get

$$X = \frac{1}{s^2+s-2} = \frac{1}{(s+2)(s-1)} \quad .$$

Since $Y = (s+1)X$, we have

$$Y = \frac{s+1}{(s+2)(s-1)} \quad .$$

So we found

$$X(s) = \frac{1}{(s+2)(s-1)} \quad , \quad Y(s) = \frac{s+1}{(s+2)(s-1)} \quad .$$

Finally, we need to find $x(t)$ and $y(t)$.

$$x(t) = \mathcal{L}^{-1}\{\frac{1}{(s+2)(s-1)}\} \quad , \quad y(t) = \mathcal{L}^{-1}\{\frac{s+1}{(s+2)(s-1)}\}$$

By partial fractions:

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \quad , \quad \frac{1}{(s+2)(s-1)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \quad .$$

So $1 = A(s-1) + B(s+2)$. When $s = 1$, $1 = B \cdot (1+2)$ so $B = \frac{1}{3}$. When $s = -2$, $1 = A(-2-1)$ so $A = -\frac{1}{3}$. So

$$X = -\frac{1}{3} \cdot \frac{1}{s+2} + \frac{1}{3} \cdot \frac{1}{s-1} \quad .$$

So

$$x(t) = \mathcal{L}^{-1}\{-\frac{1}{3}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s-1}\} = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} \quad .$$

As for $Y(s)$:

$$\frac{s+1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \quad , \quad \frac{s+1}{(s-2)(s+1)} = \frac{A(s-1)+B(s+2)}{(s+2)(s-1)} \quad .$$

So $s+1 = A(s-1) + B(s+2)$. When $s = 1$, $1+1 = B \cdot (1+2)$ so $B = \frac{2}{3}$. When $s = -2$, $-2+1 = A(-2-1)$ so $A = \frac{1}{3}$. So

$$Y = \frac{1}{3} \cdot \frac{1}{s+2} + \frac{2}{3} \cdot \frac{1}{s-1} \quad .$$

So

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{3}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s-1}\} = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \quad .$$

Ans. to 6.1: $x(t) = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t} \quad , \quad y(t) = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \quad .$

Problem 6.2: Solve the system

$$\frac{d^2x}{dt^2} = \frac{1}{2}(-x+y) \quad ,$$

$$\frac{d^2y}{dt^2} = \frac{1}{2}(x-y) \quad ,$$

$$x(0) = 0 \quad , \quad x'(0) = 2 \quad ;$$
$$y(0) = 0 \quad , \quad y'(0) = 0 \quad .$$

Solution: Let $\mathcal{L}\{x\} = X(s), \mathcal{L}\{y\} = Y(s)$. Applying $\mathcal{L}$ to both equations we get

$$\mathcal{L}\{\frac{d^2x}{dt^2}\} = -\frac{1}{2}\mathcal{L}\{x\} + \frac{1}{2}\mathcal{L}\{y\} \quad ,$$
$$\mathcal{L}\{\frac{d^2y}{dt^2}\} = \frac{1}{2}\mathcal{L}\{x\} - \frac{1}{2}\mathcal{L}\{y\} \quad ,$$

Now $\mathcal{L}\{\frac{d^2x}{dt^2}\} = s^2X(s) - x(0)s - x'(0) = s^2X(s) - 2$, $\mathcal{L}\{\frac{d^2y}{dt^2}\} = s^2Y(s) - y(0)s - y'(0) = s^2Y(s)$

So we have the system

$$s^2X - 2 = -\frac{1}{2}X + \frac{1}{2}Y$$
$$s^2Y = \frac{1}{2}X - \frac{1}{2}Y$$

Cleaning up:

$$(s^2 + \frac{1}{2})X - \frac{1}{2}Y = 2 \quad ,$$
$$(s^2 + \frac{1}{2})Y = \frac{1}{2}X \quad .$$

From the second equation we get

$$X = (2s^2 + 1)Y \quad .$$

Substituting this into the first equation:

$$(s^2 + \frac{1}{2})(2s^2 + 1)Y - \frac{1}{2}Y = 2 \quad ,$$

Factor out Y:

$$\left((s^2 + \frac{1}{2})(2s^2 + 1) - \frac{1}{2}\right)Y = 2 \quad ,$$
$$\left(2s^4 + 2s^2 + \frac{1}{2} - \frac{1}{2}\right)Y = 2 \quad ,$$
$$(2s^4 + 2s^2)Y = 2 \quad ,$$
$$2s^2(s^2 + 1)Y = 2 \quad ,$$

So

$$Y = \frac{1}{s^2(s^2 + 1)}$$

Going back to X:

$$X = \frac{2s^2 + 1}{s^2(s^2 + 1)}$$

Finally, we need to find $x(t)$ and $y(t)$.

$$x(t) = \mathcal{L}^{-1}\{\frac{2s^2+1}{s^2(s^2+1)}\} \quad , \quad y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2(s^2+1)}\}$$

By partial fractions:

$$X = \frac{2s^2+1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2+1}$$

$$\frac{2s^2+1}{s^2(s^2+1)} = \frac{(As+B)(s^2+1)+Cs^2}{s^2(s^2+1)} \quad , so$$

$$2s^2+1 = (As+B)(s^2+1)+Cs^2 = As^3+Bs^2+As+B+Cs^2 = As^3+(B+C)s^2+As+B \quad .$$

Comparing coefficients, we get:

$$A = 0 \quad , \quad B+C = 2 \quad , \quad A = 0 \quad , \quad B = 1 \quad ,$$

so $A = 0$, $B = 1$, $C = 1$ and we have

$$X = \frac{1}{s^2} + \frac{1}{s^2+1}$$

So

$$x(t) = \mathcal{L}^{-1}\{\frac{1}{s^2} + \frac{1}{s^2+1}\} = t + \sin t$$

As for $Y(s)$:

$$Y = \frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2+1}$$

$$\frac{1}{s^2(s^2+1)} = \frac{(As+B)(s^2+1)+Cs^2}{s^2(s^2+1)} \quad , so$$

$$1 = (As+B)(s^2+1)+Cs^2 = As^3+Bs^2+As+B+Cs^2 = As^3+(B+C)s^2+As+B \quad .$$

Comparing coefficients, we get:

$$A = 0 \quad , \quad B+C = 0 \quad , \quad A = 0 \quad , \quad B = 1 \quad ,$$

so $A = 0$,$B = 1$,$C = -1$ and we have

$$Y = \frac{1}{s^2} - \frac{1}{s^2+1} \quad .$$

So

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s^2} - \frac{1}{s^2+1}\} = t - \sin t \quad .$$

Ans. to 6.2: $x(t) = t + \sin t$, $y(t) = t - \sin t$.