Important Formula

Once we know the Laplace transform \( F(s) \) of some function \( f(t) \) we can immediately figure out the Laplace transform of \( t^n f(t) \) for any positive integer \( n \)

If \( F(s) = \mathcal{L}\{f(t)\} \), and \( n = 1, 2, 3, \ldots \), then

\[
\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) .
\]

Problem 4.1: Find \( \mathcal{L}\{t \cos 4t\} \)

Solution: Here \( f(t) = \cos 4t \) and \( n = 1 \), so \( F(s) = \frac{s}{s^2 + 16} \) and \( n = 1 \). The first derivative of \( F(s) \) is, by the Quotient Rule

\[
F'(s) = \frac{s'(s^2 + 16) - s(s^2 + 16)'}{(s^2 + 16)^2} = \frac{(1)(s^2 + 16) - s(2s)}{(s^2 + 16)^2} = \frac{s^2 + 16 - 2s^2}{(s^2 + 16)^2} = \frac{-s^2 + 16}{(s^2 + 16)^2} .
\]

So

\[
\mathcal{L}\{t \cos 4t\} = (-1)^1 \frac{-s^2 + 16}{(s^2 + 16)^2} = \frac{s^2 - 16}{(s^2 + 16)^2} .
\]

Ans. to 4.1: \( \mathcal{L}\{t \cos 4t\} = \frac{s^2 - 16}{(s^2 + 16)^2} . \)

Problem 4.2: Find \( \mathcal{L}\{t^2 \sin 2t\} \)

Solution: Here \( f(t) = \sin 2t \) and \( n = 2 \), so \( F(s) = \frac{2}{s^2 + 4} = 2(s^2 + 4)^{-1} \) and \( n = 2 \). The first derivative of \( F(s) \) is (by the Chain Rule)

\[
F'(s) = 2(-1)(s^2 + 4)^{-2} \cdot (2s) = \frac{-4s}{(s^2 + 4)^2} = (-4s)(s^2 + 4)^{-2} .
\]

By the Product Rule and the Chain Rule

\[
F''(s) = (-4s)'(s^2 + 4)^{-2} + (-4s)((s^2 + 4)^{-2})' = (-4)(s^2 + 4)^{-2} + (-4)(-2)(s^2 + 4)^{-3}(2s)
\]

\[
= (-4)(s^2 + 4)^{-2} + (16s^2)(s^2 + 4)^{-3} = \frac{-4}{(s^2 + 4)^2} + \frac{16s^2}{(s^2 + 4)^3} = \frac{-4(s^2 + 4) + 16s^2}{(s^2 + 4)^3}
\]

\[
= \frac{-4s^2 - 16 + 16s^2}{(s^2 + 4)^3} = \frac{12s^2 - 16}{(s^2 + 4)^3} = \frac{4(3s^2 - 4)}{(s^2 + 4)^3} .
\]

So

\[
\mathcal{L}\{t^2 \sin 2t\} = (-1)^2 \frac{4(3s^2 - 4)}{(s^2 + 4)^3} = \frac{4(3s^2 - 4)}{(s^2 + 4)^3} .
\]

Ans. to 4.2: \( \mathcal{L}\{t^2 \sin 2t\} = \frac{4(3s^2 - 4)}{(s^2 + 4)^3} . \)
Important Definition

The Convolution of two functions \( f(t) \) and \( g(t) \) written \( f * g(t) \) is defined by the integral

\[
(f * g)(t) = \int_0^t f(\tau)g(t - \tau) \, d\tau .
\]

Important Formulas

If \( f(t) \) and \( g(t) \) are nice functions on \([0, \infty)\), then

\[
\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\} = F(s)G(s) .
\]

\[
\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) .
\]

Problem 4.3: Evaluate

\[
\mathcal{L}\left\{ \int_0^t e^{2\tau} \cos(3(t - \tau)) \, d\tau \right\} .
\]

Solution. The integral is the convolution \((f * g)(t)\) of \(f(t) = e^{2t}\) and \(g(t) = \cos(3t)\). From the tables we know

\[
\mathcal{L}\{e^{2t}\} = \frac{1}{s - 2} , \quad \mathcal{L}\{\cos 3t\} = \frac{s}{s^2 + 9} .
\]

So

\[
\mathcal{L}\left\{ \int_0^t e^{2\tau} \cos(3(t - \tau)) \, d\tau \right\} = \frac{1}{s - 2} \cdot \frac{s}{s^2 + 9} = \frac{s}{(s - 2)(s^2 + 9)} .
\]

Ans. to 4.3: \( \mathcal{L}\left\{ \int_0^t e^{2\tau} \cos(3(t - \tau)) \, d\tau \right\} = \frac{s}{(s - 2)(s^2 + 9)} .
\]

Problem 4.4: Evaluate

\[
\mathcal{L}^{-1}\left\{ \frac{1}{(s + 1)(s - 1)} \right\} ,
\]

by using convolution (No credit for other methods [of course you can do it with partial fractions]).

Solution: Here

\[
F(s) = \frac{1}{s + 1} , \quad G(s) = \frac{1}{s - 1} .
\]

So

\[
f(t) = e^{-t} , \quad g(t) = e^t .
\]

\[
f * g(t) = \int_0^t e^{-\tau}e^{t - \tau} \, d\tau = \int_0^t e^{-2\tau}e^t \, d\tau = e^t \int_0^t e^{-2\tau} \, d\tau
\]

\[
= e^t \left. \frac{e^{-2\tau}}{-2} \right|_0^t = e^t \left( \frac{e^{-2t}}{-2} - \frac{e^0}{-2} \right) = \left( \frac{e^{-t}}{-2} - \frac{e^t}{-2} \right) = \frac{1}{2}(e^t - e^{-t}) .
\]

Ans. to 4.4: \( \mathcal{L}^{-1}\left\{ \frac{1}{(s + 1)(s - 1)} \right\} = \frac{1}{2}(e^t - e^{-t}) .
\]
Transform of an Integral

An important special case is when \( g(t) = 1 \) and we get

\[
\mathcal{L}\{\int_0^t f(\tau) \, d\tau\} = \frac{F(s)}{s},
\]

\[
\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) \, d\tau
\]

Problem 4.5: Compute

\[
\mathcal{L}\{\int_0^t \sin 5y \, dy\}
\]

Solution: Here \( f(t) = \sin 5t \), so \( F(s) = \frac{5}{s^2 + 25} \) and

\[
\mathcal{L}\{\int_0^t \sin 5y \, dy\} = \frac{5}{s^2 + 25} = \frac{5}{s(s^2 + 5)}.
\]

Ans. to 4.5: \( \frac{5}{s(s^2 + 5)} \).

Problem 4.6: Compute

\[
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\}
\]

without using partial fractions.

Solution: \( F(s) = \frac{1}{s^2 + 1} \), so \( f(t) = \sin t \), so

\[
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} = \int_0^t f(\tau) \, d\tau = \int_0^t \sin \tau \, d\tau = (-\cos \tau)\bigg|_0^t = (-\cos(t)) - (-\cos(0)) = 1 - \cos t.
\]

Ans. to 4.6: \( 1 - \cos t \).

Solving Integral Equations

Problem 4.7: Use Laplace Transform to solve the following integral equation

\[
f(t) + \int_0^t (t - \tau) f(\tau) \, d\tau = t.
\]

In convolution notation, this is

\[
f + t * f = t.
\]

Applying \( \mathcal{L} \) we get

\[
\mathcal{L}\{f\} + \mathcal{L}\{t * f\} = \mathcal{L}\{t\}.
\]
So:
\[ \mathcal{L}\{f\} + \mathcal{L}\{t\}\mathcal{L}\{f\} = \mathcal{L}\{t\} \]

Let, as usual \( \mathcal{L}\{f\} = F \). Recall that \( \mathcal{L}\{t\} = \frac{1}{s^2} \), so

\[ F + F \frac{1}{s^2} = \frac{1}{s^2} \]

Doing the algebra to solve \( F \):

\[ F(1 + \frac{1}{s^2}) = \frac{1}{s^2} \]
\[ F \left(1 + \frac{s^2}{s^2}\right) = \frac{1}{s^2} \]

so

\[ F = \frac{1}{1 + s^2} \]

So

\[ f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{1 + s^2} \right\} = \sin t \]

**Ans. to 4.7:** \( f(t) = \sin t \).