Dr. Z.'s Calc5 Lecture 23 Handout: Numerical Solutions of Partial Differential Equations

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Important Definitions: Discretization

The **discrete approximations** of the second derivatives with **mesh-size** h are:

$$u_{xx} \approx \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] \quad ,$$

$$u_{yy} \approx \frac{1}{h^2}[u(x,y+h) - 2u(x,y) + u(x,y-h)] \quad .$$

The **five-point approximation** of the Laplacian $u_{xx} + u_{yy}$ (in 2D) is

$$u_{xx} + u_{yy} \approx \frac{1}{h^2}[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)]$$

To **numerically** (approximately) solve the **Dirichlet** problem $u_{xx} + u_{yy} = 0$ in a region D with **boundary condition** $u(x,y) = F(x,y)$ along the boundary with mesh-size h, you set $u_{i,j} = u(ih, jh)$ and set-up a system of linear equation as follows.

For each (ih, jh) **inside** the region, you have an equation

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0 \quad ,$$

and for every **boundary point**

$$u_{i,j} = F(ih, jh) \quad .$$

Then do the linear algebra, and the solutions, $\{u_{i,j}\}$ would give you approximations for the values of the "real thing" at the interior points $\{(ih, jh)\}$.

Problem 23.1: Approximate, with mesh-size $h = 1$, the solution of the boundary-value problem

$$u_{xx} + u_{yy} = 0 \quad , \quad 0 < x < 3 \quad , \quad 0 < y < 3 \quad ;$$

subject to the boundary conditions

$$u(0,y) = 2y \quad , \quad 0 < y < 3 \quad ; \quad u(3,y) = -y \quad , \quad 0 < y < 3 \quad ;$$

$$u(x,0) = -x \quad , \quad 0 < x < 3 \quad ; \quad u(x,3) = 2x \quad , \quad 0 < x < 3 \quad .$$

Solution: There are 12 points. 8 are on the boundary, and four are inside. The boundary points are

$$P_{10} = (1,0) \quad , \quad P_{20} = (2,0) \quad ,$$

$$P_{13} = (1,3) \quad , \quad P_{23} = (2,3) \quad ,$$

$$P_{01} = (0,1) \quad , \quad P_{02} = (0,2) \quad ,$$
$$P_{31} = (3,1) \quad , \quad P_{32} = (3,2) \quad .$$

P_{10} and P_{20} are on $y = 0$, so using the data $u(x,0) = -x \quad , 0 < x < 3$, we get

$$u_{10} = u(P_{10}) = u(1,0) = -1 \quad , \quad u_{20} = u(P_{20}) = u(2,0) = -2 \quad .$$

P_{13} and P_{23} are on $y = 3$, so using the data $u(x,3) = 2x \quad , 0 < x < 3$, we get

$$u_{13} = u(P_{13}) = u(1,3) = 2 \cdot 1 = 2 \quad , \quad u_{23} = u(P_{23}) = u(2,3) = 2 \cdot 2 = 4 \quad .$$

P_{01} and P_{02} are on $x = 0$, so using the data $u(0,y) = 2y \quad , 0 < y < 3$, we get

$$u_{01} = u(P_{01}) = u(0,1) = 2 \cdot 1 = 2 \quad , \quad u_{02} = u(P_{02}) = u(0,2) = 2 \cdot 2 = 4 \quad .$$

P_{31} and P_{32} are on $x = 3$, so using the data $u(3,y) = -y \quad , 0 < y < 3$, we get

$$u_{31} = u(P_{31}) = u(3,1) = -1 \quad , \quad u_{32} = u(P_{32}) = u(3,2) = -2 \quad .$$

Summarizing, we have the following data regarding the **boundary**

$$u_{10} = -1 \quad , \quad u_{20} = -2 \quad , \quad u_{13} = 2 \quad , \quad u_{23} = 4 \quad ,$$
$$u_{01} = 2 \quad , \quad u_{02} = 4 \quad , \quad u_{31} = -1 \quad , \quad u_{32} = -2 \quad .$$

Regarding the **interior points** we have.

Point $(1,1)$:
$$u_{1+1,1} + u_{1,1+1} + u_{1-1,1} + u_{1,1-1} - 4u_{1,1} = 0 \quad ,$$
meaning
$$u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} = 0 \quad .$$

Point $(2,1)$:
$$u_{2+1,1} + u_{2,1+1} + u_{2-1,1} + u_{1,2-1} - 4u_{2,1} = 0 \quad ,$$
meaning
$$u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} = 0 \quad .$$

Point $(1,2)$:
$$u_{1+1,2} + u_{1,2+1} + u_{1-1,2} + u_{1,2-1} - 4u_{1,2} = 0 \quad ,$$

meaning

$$u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} = 0 \quad .$$

Point $(2,2)$:

$$u_{2+1,2} + u_{2,2+1} + u_{2-1,2} + u_{2,2-1} - 4u_{2,2} = 0 \quad ,$$

meaning

$$u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \quad .$$

So we have the following four linear equations:

$$u_{2,1} + u_{1,2} + u_{0,1} + u_{1,0} - 4u_{1,1} = 0 \quad ,$$

$$u_{3,1} + u_{2,2} + u_{1,1} + u_{2,0} - 4u_{2,1} = 0 \quad ,$$

$$u_{2,2} + u_{1,3} + u_{0,2} + u_{1,1} - 4u_{1,2} = 0 \quad ,$$

$$u_{3,2} + u_{2,3} + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \quad .$$

Now we have to plug-in the known values for the boundary points, namely:

$$u_{10} = -1 \quad , \quad u_{20} = -2 \quad , \quad u_{13} = 2 \quad , \quad u_{23} = 4 \quad ,$$

$$u_{01} = 2 \quad , \quad u_{02} = 4 \quad , \quad , u_{31} = -1 \quad , \quad u_{32} = -2 \quad .$$

Our system of linear equations for the unknowns $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$ becomes:

$$u_{2,1} + u_{1,2} + 2 + (-1) - 4u_{1,1} = 0 \quad ,$$

$$(-1) + u_{2,2} + u_{1,1} + (-2) - 4u_{2,1} = 0 \quad ,$$

$$u_{2,2} + (2) + (4) + u_{1,1} - 4u_{1,2} = 0 \quad ,$$

$$-2 + 4 + u_{1,2} + u_{2,1} - 4u_{2,2} = 0 \quad .$$

Moving all numbers to the right side, and writing each equation in the order $u_{1,1}, u_{2,1}, u_{1,2}, u_{2,2}$, we get the system:

$$-4u_{1,1} + u_{2,1} + u_{1,2} = -1 \quad ,$$

$$u_{1,1} - 4u_{2,1} + u_{2,2} = 3 \quad ,$$

$$u_{1,1} - 4u_{1,2} + u_{2,2} = -6 \quad ,$$

$$u_{2,1} + u_{1,2} - 4u_{2,2} = -2 \quad .$$

Or in matrix notation

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -6 \\ -2 \end{pmatrix} \quad .$$

Solving this system, either by hand, using Gaussian elimination, or using *Matlab* or *Maple*, we get the solutions:

$$u_{11} = \frac{5}{8} \quad , \quad u_{21} = -\frac{3}{8} \quad , \quad u_{12} = \frac{15}{8} \quad , \quad u_{22} = \frac{7}{8} \quad .$$

In other words, the approximations for the **interior points** are

Ans. to 23.1:

$$u(1,1) \approx \frac{5}{8} \quad , \quad u(2,1) \approx -\frac{3}{8} \quad , \quad u(1,2) \approx \frac{15}{8} \quad , \quad u(2,2) \approx \frac{7}{8} \quad .$$