Dr. Z.'s Calc5 Lecture 22 Handout: Numerical Solutions of Ordinary Differential Equations

By Doron Zeilberger

Important Method: (Euler's method for solving a first-order ode)

For the **initial value problem**

$$y' = f(x, y) \quad , \quad y(x_0) = y_0 \quad ,$$

with **mesh-size** h, you define

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \dots \quad ,$$

and compute, **one-step-at-a-time**

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad , \quad n = 1, 2 \dots .$$

y_n is an **approximation** for $y(x_n)$. The smaller h, the better the approximation.

Problem 22.1: Use the **Euler method** to find an approximate value for $y(3)$ if $y(x)$ is the solution of the initial value problem ode

$$y' = x - y + 1 \quad , \quad y(1) = 3 \quad ,$$

using mesh-size $h = \frac{1}{2}$.

Solution of 22.1:

$$x_0 = 1 \quad , \quad x_1 = 1+\frac{1}{2}\cdot 1 = \frac{3}{2} \quad , \quad x_2 = 1+\frac{1}{2}\cdot 2 = 2 \quad , \quad x_3 = 1+\frac{1}{2}\cdot 3 = \frac{5}{2} \quad , \quad x_4 = 1+\frac{1}{2}\cdot 4 = 3 \quad .$$

We need to find y_4 (since $x_4 = 3$, and we want to approximate $y(3)$).

Euler's method tells you that

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) = y_{n-1} + \frac{1}{2} \cdot (x_{n-1} - y_{n-1} + 1) \quad , \quad n = 1, 2 \dots .$$

Now

$$y_n = y_{n-1} + \frac{1}{2}(x_{n-1} - y_{n-1} + 1) = \frac{x_{n-1} + y_{n-1} + 1}{2} \quad ,$$

so

$$y_n = \frac{x_{n-1} + y_{n-1} + 1}{2} \quad .$$

Of course $y_0 = 3$ (since $y(1) = 3$). When $n = 1$ we have

$$y_1 = \frac{x_0 + y_0 + 1}{2} = \frac{1 + 3 + 1}{2} = \frac{5}{2} \quad .$$

When $n = 2$ we have

$$y_2 = \frac{x_1 + y_1 + 1}{2} = \frac{\frac{3}{2} + \frac{5}{2} + 1}{2} = \frac{5}{2} \quad .$$

When $n = 3$ we have

$$y_3 = \frac{x_2 + y_2 + 1}{2} = \frac{2 + \frac{5}{2} + 1}{2} = \frac{11}{4} \quad .$$

When $n = 4$ we have

$$y_4 = \frac{x_3 + y_3 + 1}{2} = \frac{\frac{5}{2} + \frac{11}{4} + 1}{2} = \frac{25}{8} \quad .$$

Ans. to 22.1: The approximate value of $y(3)$ (where $y(x)$ is the solution of $y' = x - y + 1, y(1) = 3$) is $\frac{25}{8} = 3.125$.

Note: In this problem, we can easily solve it explicitly (you do it!) and get $y(x) = x + 2e^{1-x}$, so the **exact** value is $y(3) = 3.27067\ldots$. Our approximation is not very good, since h was too large. If you use a computer with $h = 1/10$ you would get a much better approximation, and with $h = 1/100000$ an even better one.

Important Method (The Improved Euler's method for solving a first-order ode)

To solve the **initial value problem**

$$y' = f(x, y) \quad , \quad y(x_0) = y_0$$

with **mesh-size** h, you define

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \ldots \quad ,$$

And compute, **one-step-at-a-time**

$$y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad ,$$

$$y_n = y_{n-1} + h\frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2} \quad , \quad n = 1, 2 \ldots .$$

Then y_n is an **approximation** for $y(x_n)$. The smaller h, the better the approximation.

Problem 22.2: Use the **improved Euler method** to find an approximate value for $y(2)$ if $y(x)$ is the solution of the initial value problem ode

$$y' = x - y + 1 \quad , \quad y(1) = 3 \quad .$$

Using the mesh-size $h = \frac{1}{2}$.

Solution of 22.2: Here we only have to go to y_2.

$$x_0 = 1 \quad , \quad x_1 = 1 + \frac{1}{2} \cdot 1 = \frac{3}{2} \quad , \quad x_2 = 1 + \frac{1}{2} \cdot 2 = 2 \quad .$$

We need to find y_2 (since $x_2 = 2$, and we need to approximate $y(2)$).

The improved Euler's method tells you that

$$y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad ,$$

$$y_n = y_{n-1} + h\frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2} \quad , n = 1, 2 \ldots .$$

So in this problem, taking $h = \frac{1}{2}$ and $f(x, y) = x - y + 1$:

$$y_n^* = y_{n-1} + \frac{1}{2}(x_{n-1} - y_{n-1} + 1) = \frac{1}{2}(x_{n-1} + y_{n-1} + 1) \quad ,$$

$$y_n = y_{n-1} + \frac{x_{n-1} - y_{n-1} + 1 + x_n - y_n^* + 1}{4} = y_{n-1} + \frac{x_{n-1} - y_{n-1} + 1 + x_n - \frac{1}{2}(x_{n-1} + y_{n-1} + 1) + 1}{4}$$

$$= \frac{x_{n-1} + 5y_{n-1} + 2x_n + 3}{8} \quad .$$

(You do the algebra!) So

$$y_n = \frac{x_{n-1} + 2x_n + 5y_{n-1} + 3}{8} \quad .$$

We start with $y_0 = 3$ (the initial value). When $n = 1$, we have

$$y_1 = \frac{x_0 + 2x_1 + 5y_0 + 3}{8} = \frac{1 + 2 \cdot \frac{3}{2} + 5 \cdot 3 + 3}{8} = \frac{1 + 3 + 15 + 3}{8} = \frac{22}{8} = \frac{11}{4} \quad ,$$

$$y_2 = \frac{x_1 + 2x_2 + 5y_1 + 3}{8} = \frac{\frac{3}{2} + 2 \cdot 2 + 5 \cdot \frac{11}{4} + 3}{8} = \frac{89}{32} = 2.78125 \quad .$$

Ans. to 22.2: The approximate value of $y(2)$ is $\frac{89}{32} = 2.78125$.

Note: Once again $y(x) = x + 2e^{1-x}$, and the **exact** value is $y(2) = 2.7357\ldots$. Our approximation is not very good, since h was too large. But the unimproved Euler method, in problem 22.1 gave us: $y_2 = 2.5$, so the improved Euler method gets closer!

Important Method: Fourth-Order Runge-Kutta (RK4)

To approximate solutions of

$$y' = f(x, y) \quad , \quad y(x_0) = y_0 \quad ,$$

at $x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh$ do the following starting at y_0, for $n = 1, 2, \ldots$

$$k_1 = f(x_{n-1}, y_{n-1})$$

$$k_2 = f(x_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hk_1)$$

$$k_3 = f(x_{n-1} + \frac{1}{2}h, y_{n-1} + \frac{1}{2}hk_2)$$

$$k_4 = f(x_{n-1} + h, y_{n-1} + hk_3) \quad ,$$

and finally

$$y_n = y_{n-1} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad .$$

Problem 22.3: Use the RK4 method with $h = 0.1$ to obtain an approximation to $y(1.1)$ for the solution of $y' = 2xy, y(1) = 1$.

Solution of 22.3: Here $x_0 = 1, y_0 = 1$ and $f(x, y) = 2xy$. We only need to perform **one** step, since $x_1 = 1.1$.

$$k_1 = f(x_0, y_0) = f(1, 1) = 2 \quad ,$$

$$k_2 = f(x_0+\frac{1}{2}(0.1), y_0+\frac{1}{2}(0.1)2) = f(1+\frac{1}{2}(0.1), 1+\frac{1}{2}(0.1)2) = f(1.05, 1.1) = 2(1.05)(1.1) = 2.31$$

$$k_3 = f(x_0+\frac{1}{2}h, y_0+\frac{1}{2}hk_2) = f(1+\frac{1}{2}(0.1), 1+\frac{1}{2}(0.1)(2.31)) = f(1.05, 1.1155) = 2(1.05)(1.1155) = 2.34255$$

$$k_4 = f(x_0+h, y_0+hk_3) = f(1+0.1, 1+(0.1)2.34255) = f(1.1, 1.234255) = 2(1.1)(1.234255) = 2.715361$$

and finally

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.1}{6}(2 + 2(2.31) + 2(2.34255) + 2.715361) = 1.23367435 \quad .$$

Ans. to 22.3: $y(1.1)$ is approximately 1.23367435

First Note: In real life a computer would do it!

Second Note: In this case we can solve it **exactly** getting $y(x) = e^{x^2-1}$ (you do it!). The exact answer is $y(1.1) = e^{1.1^2-1} = e^{0.21} = 1.2336786\ldots$. **Not Bad!**.