Important Definition: The Fourier Transform and The Inverse Fourier Transform

Fourier Transform:
\[
\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x} \, dx = F(\alpha)
\]

Inverse Fourier Transform:
\[
\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x} \, d\alpha = f(x)
\]

Important Definitions: The Fourier Sine Transform and The Inverse Fourier Sine Transform

Fourier Sine Transform:
\[
\mathcal{F}_s\{f(x)\} = \int_{0}^{\infty} f(x)\sin\alpha x \, dx = F(\alpha)
\]

Inverse Fourier Sine Transform:
\[
\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_{0}^{\infty} F(\alpha)\sin \alpha x \, d\alpha = f(x)
\]

Important Definitions: The Fourier Cosine Transform and The Inverse Fourier Cosine Transform

Fourier Cosine Transform:
\[
\mathcal{F}_c\{f(x)\} = \int_{0}^{\infty} f(x)\cos \alpha x \, dx = F(\alpha)
\]

Inverse Fourier Cosine Transform:
\[
\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int_{0}^{\infty} F(\alpha)\cos \alpha x \, d\alpha = f(x)
\]

Important property of the Fourier Transform

If \(\mathcal{F}\{f(x)\} = F(\alpha)\) then for \(n = 1, 2, 3, \ldots\)
\[
\mathcal{F}\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha)
\]
Important property of the Fourier Sine Transform

If \( F_s\{f(x)\} = F(\alpha) \) then
\[
F_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0).
\]

Important property of the Fourier Cosine Transform

If \( F_c\{f(x)\} = F(\alpha) \) then
\[
F_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0).
\]

**Problem 21.1**: Solve the heat equation \( 3u_{xx} = u_t, \quad -\infty < x < \infty, \quad t > 0 \) subject to

\[
u(x,0) = \begin{cases} 4, & \text{if } |x| < 2; \\ 0, & \text{if } |x| > 2. \end{cases}
\]

**Solution of 21.1**: We are looking for a function of the two variables \( x \) (space) and \( t \) (time), called \( u(x,t) \), satisfying some conditions (a pde and initial condition). Instead we will first look for its Fourier transform, in the variable \( x \) (leaving \( t \) alone) \( F\{u(x,t)\} = U(\alpha,t) \).

Applying \( F \) to the pde yields \( F\{3u_{xx}\} = F\{u_t\} \). Using the property of \( F \) that \( F\{f^{(n)}(x)\} = (-i\alpha)^n F(\alpha) \), we get

\[
3(i\alpha)^2 U(\alpha,t) = U(\alpha,t) t,
\]
so
\[
-3\alpha^2 U(\alpha,t) = \frac{d}{dt} U(\alpha,t).
\]
We got the ode, in the variable \( t \):
\[
\frac{dU}{dt} + 3\alpha^2 U = 0.
\]
Solving this simple ode, gives
\[
U(\alpha,t) = ce^{-3\alpha^2 t},
\]
where the constant \( c \) is \( U(\alpha,0) \). \( U(\alpha,0) \) is the Fourier transform of the function describing \( u(x,0) \) namely of

\[
f(x) = \begin{cases} 4, & \text{if } |x| < 2; \\ 0, & \text{if } |x| > 2. \end{cases}
\]

The next task is to compute the Fourier transform of this \( f(x) \).
\[
F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{i\alpha x} \, dx = \int_{-2}^{2} 4e^{i\alpha x} \, dx = \left. 4 \frac{e^{i\alpha x}}{i\alpha} \right|_{-2}^{2} = 4 \frac{e^{2i\alpha} - e^{-2i\alpha}}{i\alpha} = 8 \frac{e^{2i\alpha} - e^{-2i\alpha}}{2i\alpha} = 8 \frac{\sin 2\alpha}{\alpha}
\]
So \( c \) above is \( 8 \frac{\sin 2\alpha}{\alpha} \), and we get
\[
U(\alpha,t) = 8 \frac{\sin 2\alpha}{\alpha} e^{-3\alpha^2 t}
\]
To go back to \( u(x, t) \) we apply \( \mathcal{F}^{-1} \)

\[
\begin{align*}
\mathcal{F}^{-1}\{U(\alpha, t)\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\alpha, t)e^{-i\alpha x} \, d\alpha = \frac{4}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2\alpha}{\alpha} e^{-3\alpha^2 t}e^{-i\alpha x} \, d\alpha.
\end{align*}
\]

This is a correct answer, but, using \( e^{-i\alpha x} = \cos(\alpha x) - i \sin(\alpha x) \), and seeing that \( \int_{-\infty}^{\infty} \frac{\sin 2\alpha}{\alpha} e^{-3\alpha^2 t} \sin \alpha x \, d\alpha = 0 \), since the sine function is an odd function, we get the simpler solution (without any complex numbers!)

\[
\mathcal{F}^{-1}\{U(\alpha, t)\} = \frac{4}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2\alpha \cos \alpha x}{\alpha} e^{-3\alpha^2 t} \, d\alpha.
\]

**Ans. to 21.1:** \( u(x, t) = \frac{4}{\pi} \int_{-\infty}^{\infty} \frac{\sin 2\alpha \cos \alpha x}{\alpha} e^{-3\alpha^2 t} \, d\alpha \).

**Problem 21.2:** Solve the pde

\[
u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad y > 0
\]

subject to the boundary conditions

\[
\begin{align*}
u(0, y) &= 0 , \quad u(\pi, y) = 3e^{-2y} , \quad y > 0 ; \\
u_y(x, 0) &= 0 , \quad 0 < x < \pi .
\end{align*}
\]

**Solution:** Instead of looking for \( u(x, t) \) we will look for its Fourier cosine transform with respect to the variable \( y \), leaving \( x \) alone, \( \mathcal{F}_c\{u(x, y)\} = U(x, \alpha) \). Applying \( \mathcal{F}_c \) to the pde gives

\[
\mathcal{F}_c\{u_{xx}\} + \mathcal{F}_c\{u_{yy}\} = \mathcal{F}_c\{0\} \quad 0 < x < \pi .
\]

Since \( \mathcal{F}_c\{u_{yy}\} = -\alpha^2 U(x, \alpha) - u_y(x, 0) \), and the last boundary condition says that \( u_y(x, 0) = 0 \), we have \( \mathcal{F}_c\{u_{yy}\} = -\alpha^2 U(x, \alpha) \). Of course \( \mathcal{F}_c\{u_{xx}\} = U_{xx}(x, \alpha) \). So

\[
\frac{d^2U}{dx^2} - \alpha^2 U = 0 \quad 0 < x < \pi .
\]

The general solution of this ode is

\[
U(x, \alpha) = c_1 \cosh \alpha x + c_2 \sinh \alpha x .
\]

We need to translate the boundary conditions \( u(0, y) = 0, u(\pi, y) = 3e^{-2y} \) from the \( u \)-language to the \( U \)-language, by applying \( \mathcal{F}_c \).

\[
U(0, \alpha) = \mathcal{F}_c\{0\} = 0 , \quad U(\pi, \alpha) = 3\mathcal{F}_c\{e^{-2y}\} .
\]

Now, from Maple, or from a table of integrals:

\[
U(\pi, \alpha) = \mathcal{F}_c\{3e^{-2y}\} = \int_0^{\infty} 3e^{-2y} \cos \alpha y \, dy = \frac{6}{4 + \alpha^2} .
\]
We have the system of two equations and two unknowns

\[ U(0, \alpha) = c_1 \cosh 0 + c_2 \sinh 0 , \]

\[ U(\pi, \alpha) = c_1 \cosh \alpha \pi + c_2 \sinh \alpha \pi . \]

So

\[ 0 = c_1 , \]

\[ \frac{6}{4 + \alpha^2} = c_1 \cosh \alpha \pi + c_2 \sinh \alpha \pi . \]

and we get \( c_1 = 0 \) and \( c_2 = \frac{6}{(4 + \alpha^2) \sinh \alpha \pi} \), establishing that

\[ U(x, \alpha) = \frac{6 \sinh \alpha x}{(4 + \alpha^2) \sinh \alpha \pi} . \]

Applying \( \mathcal{F}^{-1} \) we get

\[ u(x, y) = \frac{12}{\pi} \int_0^{\infty} \frac{\sinh \alpha x}{(4 + \alpha^2) \sinh \alpha \pi} \cos \alpha y \, d\alpha . \]

**Ans. to 21.2:** \[ u(x, y) = \frac{12}{\pi} \int_0^{\infty} \frac{\sinh \alpha x}{(4 + \alpha^2) \sinh \alpha \pi} \cos \alpha y \, d\alpha . \]

**Note:** You have to be flexible! Here the “active” variable was \( y \), not the usual \( x \), so the formulas for \( \mathcal{F}_c \) and \( \mathcal{F}^{-1}_c \) had to be adjusted accordingly.

**Note:** If the boundary condition would have given \( u(x, 0) \) rather than \( u_y(x, 0) \) then one should use the Sine Fourier Transform.