Dr. Z.’s Calc5 Lecture 2 Handout: The Inverse Laplace Transform and Derivatives

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**Theory:** The Laplace Transform is a dictionary that goes from functions of $t$ (usually time) to functions of $s$. It is often necessary to be able to translate back! Taking the little table of Lecture 1, and reversing it we get:

(a) $\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1$

(b) $\mathcal{L}^{-1}\left\{ \frac{1}{s^k} \right\} = \frac{t^{k-1}}{(k-1)!} \quad (k = 1, 2, 3, ...)$

(c) $\mathcal{L}^{-1}\left\{ \frac{1}{s-a} \right\} = e^{at}$

(d) $\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + k^2} \right\} = \frac{\sin kt}{k}$

(e) $\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + k^2} \right\} = \frac{\cos kt}{k}$

(f) $\mathcal{L}^{-1}\left\{ \frac{1}{s^2 - k^2} \right\} = \frac{\sinh kt}{k}$

(g) $\mathcal{L}^{-1}\left\{ \frac{s}{s^2 - k^2} \right\} = \cosh kt$

Note that except for (a), each of these formulas contains infinitely many facts, since they involve parameters. For example thanks to (b) we know the inverse-Laplace-Transform of $1/s$, $1/s^2$, $1/s^3$ etc.

**Problem 2.1:** Find $\mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{5}{s-5} \right\}$

**Solution:** By linearity:

$$\mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{5}{s-5} \right\} = 3\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} + 5\mathcal{L}^{-1}\left\{ \frac{1}{s-5} \right\}.$$

By (a) (or (b) with $k = 1$)

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = 1.$$

By (c)

$$\mathcal{L}^{-1}\left\{ \frac{1}{s-5} \right\} = e^{5t}.$$

Combining, we have

$$\mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{5}{s-5} \right\} = 3\mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} + 5\mathcal{L}^{-1}\left\{ \frac{1}{s-5} \right\} = 3 \cdot 1 + 5e^{5t} = 3 + 5e^{5t}.$$

**Ans. to 2.1:** $\mathcal{L}^{-1}\left\{ \frac{3}{s} + \frac{5}{s-5} \right\} = 3 + 5e^{5t}.$
Problem 2.2: Find $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+9}\right\}$

Sol.: Remember, we can always break-up the numerator!

\[
\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}.
\]

By (e) and (d) with $k = 3$, this equals

\[
2 \cos 3t + \frac{1}{3} \sin 3t.
\]

Ans. to 2.2: $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2+9}\right\} = 2 \cos 3t + \frac{1}{3} \sin 3t$.

When we get complicated rational functions of $s$, we need to do a partial fraction decomposition.

Problem 2.3: Evaluate $\mathcal{L}^{-1}\left\{\frac{3s-4}{s^2-3s+2}\right\}$

Sol.: First we must factorize the denominator (unless it is already factored)

\[
\frac{3s-4}{s^2-3s+2} = \frac{3s-4}{(s-1)(s-2)}
\]

Now are are looking for magic numbers, let’s call them $A$ and $B$, such that

\[
\frac{3s-4}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.
\]

Adding the fractions on the right (using $a/b + c/d = (ad + bc)/bd$):

\[
\frac{3s-4}{(s-1)(s-2)} = \frac{A(s-2) + B(s-1)}{(s-1)(s-2)}.
\]

The denominators automatically match, but to make this come true, we need the numerators to match:

\[
3s-4 = A(s-2) + B(s-1).
\]

Now we plug-in convenient values. When $s = 2$ we get

\[
3 \cdot 2 - 4 = A(2-2) + B(2-1) = 0 + B = B.
\]

So $B = 2$. When $s = 1$ we get:

\[
3 \cdot 1 - 4 = A(1-2) + B(1-1).
\]

So

\[-1 = -A\]

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and so \( A = 1 \). Once we know what \( A \) and \( B \) are we go back and write:

\[
\frac{3s - 4}{(s - 1)(s - 2)} = \frac{1}{s - 1} + \frac{2}{s - 2} .
\]

So far it was just algebra. Only now are we ready to apply \( L^{-1} \).

\[
L^{-1}\left\{ \frac{3s - 4}{s^2 - 3s + 2} \right\} = L^{-1}\left\{ \frac{1}{s - 1} \right\} + L^{-1}\left\{ \frac{2}{s - 2} \right\} = L^{-1}\left\{ \frac{1}{s - 1} \right\} + 2L^{-1}\left\{ \frac{1}{s - 2} \right\} .
\]

Using (c) with \( a = 1 \) and \( a = 2 \), we get:

\[
e^t + 2e^{2t} .
\]

Ans. to 2.3:

\[
L^{-1}\left\{ \frac{3s - 4}{s^2 - 3s + 2} \right\} = e^t + 2e^{2t} .
\]

More Theory: The beauty of \( L \) is that it turns differential equations into algebraic equations!

If \( y(t) \) is any function of time, and if \( Y(s) \) is its Laplace Transform:

\[
L\{y(t)\} = Y(s) ,
\]

then the Laplace Transform of the first derivative, \( y'(t) \), is “almost” \( Y(s) \) multiplied by \( s \), and if \( y(0) = 0 \) then it is exactly that. We have:

\[
L\{y'(t)\} = sY(s) - y(0)
\]

(Prove this!, Hint: Integration by parts.) Applying this very same formula to \( y''(t) \) we have

\[
L\{y''(t)\} = s(sY(s) - y(0)) - y'(0) = s^2Y(s) - sy(0) - y'(0) .
\]

One more time:

\[
L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0) ,
\]

and so on. In general:

\[
L\{y^{(n)}(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0) .
\]

Equipped with this, we can solve Initial Value Problems.

**Problem 2.4**: Use Laplace Transform to solve the following initial-value problem.

\[
y' + 2y = e^t , \quad y(0) = -3 .
\]

**Solution**: Let \( Y(s) = L\{y(t)\} \). Applying \( L \):

\[
L\{y' + 2y\} = L\{e^t\} .
\]
So
\[ \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\} . \]

But \( \mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 3. \) Also \( \mathcal{L}\{e^t\} = \frac{1}{s-1}. \) So we have the algebraic equation:

(Let’s abbreviate, and write \( Y \) for \( Y(s) \))

\[ sY + 3 + 2Y = \frac{1}{s-1} . \]

Solving for \( Y \), we get,

\[ (s + 2)Y = \frac{1}{s-1} - 3 = \frac{4 - 3s}{s-1} . \]

Dividing both sides by \((s + 2)\), we have an explicit expression for \( Y \):

\[ Y(s) = \frac{4 - 3s}{(s - 1)(s + 2)} . \]

In order to find \( y(t) \) we need to compute

\[ \mathcal{L}^{-1}\{\frac{4 - 3s}{(s - 1)(s + 2)}\} . \]

Doing partial fractions:

\[ Y(s) = \frac{4 - 3s}{(s - 1)(s + 2)} = \frac{A}{s - 1} + \frac{B}{s + 2} . \]

We get

\[ \frac{4 - 3s}{(s - 1)(s + 2)} = \frac{A(s + 2) + B(s - 1)}{(s - 1)(s + 2)} . \]

So

\[ 4 - 3s = A(s + 2) + B(s - 1) . \]

When \( s = 1 \), we get \( 1 = 3A \) so \( A = \frac{1}{3} \) and when \( s = -2 \) we get \( 10 = B(-3) \) so \( B = -\frac{10}{3} \), so the simplified expression for \( Y \) is:

\[ Y(s) = \frac{1}{3} + \frac{-10}{s + 2} . \]

Now we are ready to take \( \mathcal{L}^{-1} \)

\[ y(t) = \mathcal{L}^{-1}\{\frac{1}{3(s - 1)} + \frac{-10}{s + 2}\} = \frac{1}{3}\mathcal{L}^{-1}\{\frac{1}{s - 1}\} + (-\frac{10}{3})\mathcal{L}^{-1}\{\frac{1}{s + 2}\} = \frac{1}{3}e^t - \frac{10}{3}e^{-2t} . \]

Ans. to 2.4: \( y(t) = \frac{1}{3}e^t - \frac{10}{3}e^{-2t} . \)

Problem 2.5: Use Laplace Transform to solve the following initial-value problem.

\[ y'' - 3y' + 2y = e^{3t} , \quad y(0) = 0 , \quad y'(0) = 0 . \]
**Solution:** As usual $\mathcal{L}\{y(t)\} = Y(s)$. Applying $\mathcal{L}$:

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{e^{3t}\}, \quad y(0) = 0, \quad y'(0) = 0.$$ 

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{s-3}.$$ 

Note that $\mathcal{L}\{y'\} = sY - y(0) = sY$, $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0) = s^2Y$, so

$$s^2Y - 3sY + 2Y = \frac{1}{s-3}. $$

$$(s^2 - 3s + 2)Y = \frac{1}{s-3}. $$

$$(s - 1)(s - 2)Y = \frac{1}{s-3}. $$

Solving for $Y$:

$$Y = \frac{1}{(s-1)(s-2)(s-3)}. $$

**Partial Fractions:**

$$\frac{1}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}. $$

$$1 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2). $$

Conveninet values: $s = 1$: $1 = A(-1)(-2)$, so $A = \frac{1}{2}$; $s = 2$: $1 = B(1)(-1)$, so $B = -1$; $s = 3$: $1 = C(2)(1)$, so $C = \frac{1}{2}$. So

$$Y = \frac{1}{s-1} + \frac{-1}{s-2} + \frac{1}{2}\frac{1}{s-3}. $$

Finally,

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} + \frac{-1}{s-2} + \frac{1}{2}\frac{1}{s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}. $$

**Ans. to 2.5:** $y(t) = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$. 

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