Important Problem (Laplace’s Equation in a Rectangle)

Solve

\[ u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \]

subject to various types of boundary conditions, involving the function itself or its derivatives on the four sides.

Problem 17.1 Solve

\[ u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \]

subject to

\[ u_x(0,y) = 0, \quad u_x(\pi,y) = 0, \quad 0 < y < 1; \]
\[ u(x,0) = 0, \quad u(x,1) = f(x), \quad 0 < x < \pi. \]

Solution: We first look for separable solutions of the type

\[ u(x,y) = X(x)Y(y). \]

Since

\[ u_{xx} = X''(x)Y(y), \quad u_{yy} = X(x)Y''(y), \]

we have

\[ X''(x)Y(y) + X(x)Y''(y) = 0. \]

Dividing by \( X(x)Y(y) \), we have

\[ \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0, \]

so

\[ \frac{X''(x)}{X(x)} = -\lambda, \quad \frac{Y''(y)}{Y(y)} = -\lambda. \]

The left side does not depend on \( y \), and the right side does not depend on \( x \), since they are the same, neither of them depends on \( x \) or \( y \), so they are both equal to the same constant, let’s call it \( -\lambda \). We have

\[ \frac{X''(x)}{X(x)} = -\lambda, \]
\[ \frac{Y''(y)}{Y(y)} = -\lambda. \]

Leading to two odes:

\[ X''(x) + \lambda X(x) = 0, \]
\[ Y''(y) - \lambda Y(y) = 0. \]
Now it is time to look at the homogeneous boundary conditions (those whose right hand side is 0).
Since \( u(x, y) = X(x)Y(y), \ u_x(x, y) = X'(x)Y(y) \), and
\[
u_x(0, y) = 0 , \quad 0 < y < 1 ,
\]
means
\[
X'(0)Y(y) = 0 .
\]
Since the function \( Y(y) \) better not be zero (or else we get the trivial, zero, solution), we must have:
\[
X'(0) = 0 .
\]
Another boundary condition is:
\[
u_x(\pi, y) = 0 , \quad 0 < y < 1 ,
\]
means
\[
X'(\pi)Y(y) = 0 ,
\]
so
\[
X'(\pi) = 0 .
\]
For future reference, \( u(x, 0) = 0 \) means
\[
X(x)Y(0) = 0 ,
\]
so \( Y(0) = 0 \).

We first have to solve the \textbf{Sturm-Liouville} system
\[
X''(x) + \lambda X(x) = 0 , \quad 0 < x < \pi , \quad X'(0) = 0 , \quad X'(\pi) = 0 .
\]

\textbf{Case I}: \( \lambda < 0 \). Writing \( \lambda = -\alpha^2 \), we get
\[
X''(x) - \alpha^2 X(x) = 0 , \quad 0 < x < \pi , \quad X'(0) = 0 , \quad X'(\pi) = 0 .
\]
So
\[
X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} ,
\]
entailing that
\[
X'(x) = c_1 \alpha e^{\alpha x} - c_2 \alpha e^{-\alpha x} ,
\]
that in turn, lead to:
\[
X'(0) = c_1 \alpha - c_2 \alpha , \quad X'(\pi) = c_1 \alpha e^{\alpha \pi} - c_2 \alpha e^{-\alpha \pi} .
\]
We have to find real numbers \( c_1, c_2 \) such that
\[
c_1 \alpha - c_2 \alpha = 0 , \quad c_1 \alpha e^{\alpha \pi} - c_2 \alpha e^{-\alpha \pi} = 0 .
\]
From the first equation $c_1 = c_2$ (since $\alpha \neq 0$), so $c_1\alpha e^{\alpha \pi} - c_1\alpha e^{-\alpha \pi} = 0$ so $c_1(\alpha e^{\alpha \pi} - \alpha e^{-\alpha \pi}) = 0$ and we get $c_1 = 0$, and hence also $c_2 = 0$, so we only got the trivial solution, that does not count.

**Case II:** $\lambda = 0$.

$$X''(x) = 0, \quad 0 < x < \pi, \quad X'(0) = 0, \quad X'(\pi) = 0.$$ 

So $X(x) = c_1 + c_2 x$, $X'(x) = c_2$, $X'(0) = c_2$, $X'(\pi) = c_2$, so $c_2 = 0$, and we got that $X(x) = c_1$ is a solution. The counterpart ode for $Y(y)$ is $Y''(y) = 0$ whose general solution is $Y(y) = c_3 + c_4 y$, so $u(x, y) = X(x)Y(y) = c_1(c_3 + c_4 y)$. Using $u(x, 0) = 0$ gives $c_3 = 0$ so $u(x, y) = c_1 c_3 y$. Renaming $c_1 c_3$, $A_0$, we get only one solution from Case II $u(x, y) = A_0 y$.

**Case III:** $\lambda > 0$. Writing $\lambda = \alpha^2$, we get

$$X''(x) + \alpha^2 X(x) = 0, \quad 0 < x < \pi, \quad X'(0) = 0, \quad X'(\pi) = 0.$$ 

So $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$, $X'(x) = -\alpha c_1 \sin(\alpha x) + \alpha c_2 \cos(\alpha x)$, $X'(0) = \alpha c_2$, $X'(\pi) = -\alpha c_1 \sin(\alpha \pi) + \alpha c_2 \cos(\alpha \pi)$. So $c_2 = 0$ (since $\alpha \neq 0$), and $\sin(\alpha \pi) = 0$.

We have to solve the equation in $\alpha$,

$$\sin(\alpha \pi) = 0.$$ 

There are infinitely many solutions $\alpha = 1, 2, \ldots$, in general $\alpha = n$ for any positive integer, (the case $\alpha = 0$ we already have from above and besides in case II we assume $\alpha > 0$). The corresponding solution is

$$X(x) = c_1 \cos(nx), \quad n = 1, 2, \ldots.$$ 

We now need to find the counterpart $Y(y)$ for each of $\lambda = n^2$ (for $n = 1, 2, 3, \ldots$).

$$Y''(y) - n^2 Y(y) = 0.$$ 

The general solution is

$$Y(y) = c_3 \sinh ny + c_4 \cosh ny.$$ 

Remember $u(x, y) = X(x)Y(y)$, so

$$u(x, y) = c_1 \cos(nx)(c_3 \sinh(ny) + c_4 \cosh(ny))$$.

Using the third boundary conditions $u(x, 0) = 0$, we get

$$0 = u(x, 0) = c_1 \cos(nx)(c_3 \sinh 0 + c_4 \cosh 0) = c_1 c_4 \cos(nx),$$

so $c_4 = 0$ and

$$u(x, y) = c_1 c_3 \cos(nx) \sinh(ny).$$
We rename \( c_1 c_3 \) to \( A_n \) and have the solution
\[
    u(x, y) = A_n \cos(nx) \sinh(ny)
\]
for an arbitrary constant \( A_n \).

Now it is time to take care of the last boundary condition, \( u(x, 1) = f(x) \). If we are lucky, and \( f(x) \) happens to be exactly of the form
\[
    f(x) = C \cos(nx)
\]
for some specific integer \( n \), and some specific constant \( C \), then
\[
    u(x, 1) = A_n \cos(nx) \sinh(n) = C \cos(nx)
\]
and we get \( A_n = C / \sinh(n) \), and the final answer would have been
\[
    u(x, y) = \frac{C}{\sinh(n)} \cos(nx) \sinh(ny)
\]
Alas, for a general \( f(x) \) that is not a constant multiple of a pure cosine function, we must go on and use the principle of superposition and Fourier series.

By the principle of superposition, the following function
\[
    u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny)
\]
is a solution to the pde plus the first three boundary conditions, for every choice of constants \( A_0, A_1, A_2 \ldots \), so without the last boundary condition, there are \( \infty \) answers. Now it is time to impose the fourth boundary condition
\[
    u(x, 1) = f(x)
\]
So
\[
    f(x) = u(x, 1) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(n) = A_0 + \sum_{n=1}^{\infty} (A_n \sinh(n)) \cos(nx)
\]
So we need the Fourier cosine series of Lecture 9.
\[
    f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx
\]
where
\[
    a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx
\]
\[
    a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx
\]
So, comparing to the situations that we have now

\[ 2A_0 = \frac{2}{\pi} \int_0^\pi f(x) \, dx, \]
\[ A_n \sinh(n) = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx, \]

Doing the algebra, gives

\[ A_0 = \frac{1}{\pi} \int_0^\pi f(x) \, dx \]
\[ A_n = \frac{2}{\pi \sinh(n)} \int_0^\pi f(x) \cos nx \, dx. \]

So the final answer is that the solution \( u(x, y) \) of our boundary-value problem is

\[ u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny), \]

where

\[ A_0 = \frac{1}{\pi} \int_0^\pi f(x) \, dx, \quad A_n = \frac{2}{\pi \sinh(n)} \int_0^\pi f(x) \cos nx \, dx. \]

This is the answer. Another way of writing the answer is the following hairy formula

\textbf{Ans. to Problem 17.1:}

\[ u(x, y) = \left( \frac{1}{\pi} \int_0^\pi f(x) \, dx \right) y + \sum_{n=1}^{\infty} \left( \frac{2}{\pi \sinh(n)} \int_0^\pi f(x) \cos nx \, dx \right) \cos(nx) \sinh(ny). \]

\textbf{Important Property: Boundary Superposition Principle}

If you have a complicated so-called \textit{Dirichlet} boundary value problem

\[ u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b \]
\[ u(0, y) = F(y), \quad u(a, y) = G(y), \quad 0 < y < b \]
\[ u(x, 0) = f(x), \quad u(x, b) = g(x), \quad 0 < x < a. \]

You break it up into two problems:

\textbf{First Problem:} Find the solution, let’s call it \( u_1(x, y) \) satisfying

\[ (u_1)_{xx} + (u_1)_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b \]
\[ u_1(0, y) = 0, \quad u_1(a, y) = 0, \quad 0 < y < b \]
\[ u_1(x, 0) = f(x), \quad u_1(x, b) = g(x), \quad 0 < x < a. \]
**Second Problem:** Find the solution, let’s call it \( u_2(x, y) \) satisfying

\[
(u_2)_{xx} + (u_2)_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b
\]

\[
u_2(0, y) = F(y), \quad u_2(a, y) = G(y), \quad 0 < y < b.
\]

\[
u_2(x, 0) = 0, \quad u_2(x, b) = 0, \quad 0 < x < a.
\]

Once you solved these (already complicated!) two problems, the **final** solution, to the original problem, is simply

\[
u(x, y) = u_1(x, y) + u_2(x, y).
\]

In other words, just add them up!