Dr. Z.'s Calc5 Lecture 16 Handout: The Wave Equation

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Important Problem

The Wave equation for the displacement, $u(x,t)$, of a sting of length L is, held tight at both ends on the interval $[0,L]$ of the x-axis, is:

$$a^2 u_{xx} = u_{tt} \quad , \quad 0 < x < L \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad 0 < x < L \quad .$$

(a is a constant that comes from the physics).

Important Special Case The Wave equation for the displacement, $v(x,t)$, of a sting of length π is:

$$a^2 v_{xx} = v_{tt} \quad ,0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$v(0,t) = 0 \quad , \quad v(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$v(x,0) = f(x) \quad , \quad v_t(x,0) = g(x) \quad , \quad 0 < x < \pi \quad .$$

(a is a constant that comes from the physics).

How to go from the general case to the special case Define

$$v(x,t) = u(\frac{L}{\pi}x, t) \quad .$$

Then

$$v_{xx} = (\frac{L}{\pi})^2 u_{xx} \quad ; \quad v(x,0) = u(\frac{L}{\pi}x, 0) = f(\frac{L}{\pi}x) \quad ; \quad v_t(x,0) = u_t(\frac{L}{\pi}x, 0) = g(\frac{L}{\pi}x) \quad ,$$

and we get the problem

$$(a\frac{\pi}{L})^2 v_{xx} = v_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0$$

$$v(0,t) = 0 \quad , \quad v(\pi,t) = 0 \quad , \quad t > 0$$

$$v(x,0) = f(\frac{L}{\pi}x) \quad , \quad v_t(x,0) = g(\frac{L}{\pi}x) \quad , \quad 0 < x < \pi.$$

Important Solution (Special case $L = \pi$) The solution of the boundary value wave equation

$$a^2 v_{xx} = v_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$v(0,t) = 0 \quad , \quad v(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$v(x,0) = f(x) \quad , \quad v_t(x,0) = g(x) \quad , \quad 0 < x < \pi \quad .$$

is

$$v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(nat) + B_n \sin(nat)\right) \sin(nx) \quad ,$$

where the numbers A_n and B_n are given by the formulas

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$B_n = \frac{2}{n\pi a} \int_0^{\pi} g(x) \sin nx \, dx.$$

Doing the above reduction, we get

Important Solution (General Case)

The solution of the boundary value wave equation

$$a^2 u_{xx} = u_{tt} \quad , \quad 0 < x < L \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = f(x) \quad , \quad u_t(x,0) = g(x) \quad , \quad 0 < x < L \quad .$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n \cos(\frac{n\pi a}{L}t) + B_n \sin(\frac{n\pi a}{L}t)\Big) \sin(\frac{n\pi}{L}x) \quad ,$$

where the numbers A_n and B_n are given by the formulas

$$A_n = \frac{2}{L} \int_0^{L} f(x) \sin \frac{n\pi}{L}x \, dx \quad ,$$

$$B_n = \frac{2}{n\pi a} \int_0^{L} g(x) \sin \frac{n\pi}{L}x \, dx.$$

Problem 16.1 Solve the boundary value pde problem:

$$u_{xx} = u_{tt} \quad , 0 < x < \pi \quad , \quad t > 0$$

$$u(0,t) = 0 \quad , \quad u(\pi,t) = 0 \quad , \quad t > 0$$

$$u(x,0) = x(\pi - x) \quad , \quad u_t(x,0) = 2x(\pi - x) \quad , \quad 0 < x < \pi$$

Solution Here $a = 1$ so:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(nt) + B_n \sin(nt)) \sin(nx) \quad ,$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx$$

$$B_n = \frac{2}{n\pi}\int_0^\pi 2x(\pi - x)\sin nx\,dx.$$

Using Maple, or integration tables,

$$\int_0^\pi x(\pi - x)\sin nx\,dx = \frac{2(1-(-1)^n)}{n^3}$$

So

$$A_n = \frac{2}{\pi}\cdot\frac{2(1-(-1)^n)}{n^3} = \frac{4(1-(-1)^n)}{\pi n^3} \quad ,$$

$$B_n = \frac{2}{n\pi}\cdot 2\cdot\frac{2(1-(-1)^n)}{n^3} = \frac{8(1-(-1)^n)}{\pi n^4} \quad .$$

Combining everything, we get:

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{4(1-(-1)^n)}{\pi n^3}\cos(nt) + \frac{8(1-(-1)^n)}{\pi n^4}\sin(nt)\right)\sin(nx) \quad .$$

This is the **answer**.

Problem 16.2 Solve the boundary value problem

$$u_{xx} = u_{tt} \quad , \quad 0 < x < 4\pi \quad , \quad t > 0 \quad ;$$

$$u(0,t) = 0 \quad , \quad u(4\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$u(x,0) = x(4\pi - x) \quad , \quad u_t(x,0) = 2x(4\pi - x) \quad , \quad 0 < x < 4\pi \quad .$$

by first solving a similar problem for the interval $[0,\pi]$, and then going back.

Solution. We have to "squeeze" the interval $[0,4\pi]$ by a ratio of 4, so we **define** a **new function**

$$v(x,t) = u(4x,t) \quad .$$

So $v_{xx} = 4^2 u_{xx}$ and $v(x,0) = u(4x,0) = 4x(4\pi - 4x) = 16x(\pi - x)$, $v_t(4x,0) = 2(4x)(4\pi - 4x) = 32x(\pi - x)$.

We have the new problem

$$(\frac{1}{4})^2 v_{xx} = v_{tt} \quad , \quad 0 < x < \pi \quad , \quad t > 0 \quad ;$$

$$v(0,t) = 0 \quad , \quad v(\pi,t) = 0 \quad , \quad t > 0 \quad ;$$

$$v(x,0) = 16x(\pi - x) \quad , \quad v_t(x,0) = 32(\pi - x) \quad , \quad 0 < x < \pi \quad .$$

Using the ready-made formula we have (here $a = \frac{1}{4}$)

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(nt/4) + B_n \sin(nt/4)) \sin(nx) \quad ,$$

where the numbers A_n and B_n are given by the formulas

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad ,$$

$$B_n = \frac{2}{n\pi(1/4)} \int_0^{\pi} g(x) \sin nx \, dx.$$

We have

$$f(x) = 16x(\pi - x) \quad , \quad g(x) = 32x(\pi - x) \quad .$$

So

$$A_n = \frac{2}{\pi} \int_0^{\pi} 16x(\pi - x) \sin nx \, dx = \frac{32}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$B_n = \frac{256}{n\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx.$$

Using Maple, or integration tables,

$$\int_0^{\pi} x(\pi - x) \sin nx \, dx = \frac{2(1-(-1)^n)}{n^3}$$

So

$$A_n = \frac{32}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx = \frac{32}{\pi} \cdot \frac{2(1-(-1)^n)}{n^3} = \frac{64(1-(-1)^n)}{\pi n^3} \quad ,$$

$$B_n = \frac{256}{n\pi} \cdot \frac{2(1-(-1)^n)}{n^3} = \frac{512(1-(-1)^n)}{\pi n^4} \quad .$$

Combing, we get:

$$v(x,t) = \sum_{n=1}^{\infty} \left(\frac{64(1-(-1)^n)}{\pi n^3} \cos(nt/4) + \frac{512(1-(-1)^n)}{\pi n^4} \sin(nt/4) \right) \sin(nx) \quad .$$

Finally, we must go back to $u(x,t)$, using

$$u(x,t) = v(x/4, t) \quad .$$

So:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{64(1-(-1)^n)}{\pi n^3} \cos(nt/4) + \frac{512(1-(-1)^n)}{\pi n^4} \sin(nt/4) \right) \sin(nx/4) \quad .$$

Since $(1) - (-1)^n$ is 0 when n is even, and is 2 when n is odd, we can write this even better as:

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{128}{\pi(2k+1)^3} \cos((2k+1)t/4) + \frac{1024}{\pi(2k+1)^4} \sin((2k+1)t/4) \right) \sin((2k+1)x/4) \quad .$$

Answer to 16.2:

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{128}{\pi(2k+1)^3} \cos((2k+1)t/4) + \frac{1024}{\pi(2k+1)^4} \sin((2k+1)t/4) \right) \sin((2k+1)x/4) \quad .$$