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**Important Problem**

The Wave equation for the displacement, \( u(x, t) \), of a sting of length \( L \) is, held tight at both ends on the interval \([0, L]\) of the \( x \)-axis, is:

\[
a^2 u_{xx} = u_{tt} \quad 0 < x < L \quad t > 0 ;
\]

\[
u(0, t) = 0 \quad u(L, t) = 0 \quad t > 0 ;
\]

\[
u(x, 0) = f(x) \quad u_t(x, 0) = g(x) \quad 0 < x < L .
\]

\((a \text{ is a constant that comes from the physics)\).\)

**Important Special Case**

The Wave equation for the displacement, \( v(x, t) \), of a sting of length \( \pi \) is:

\[
a^2 v_{xx} = v_{tt} \quad 0 < x < \pi \quad t > 0 ;
\]

\[
v(0, t) = 0 \quad v(\pi, t) = 0 \quad t > 0 ;
\]

\[
v(x, 0) = f(x) \quad v_t(x, 0) = g(x) \quad 0 < x < \pi .
\]

\((a \text{ is a constant that comes from the physics)\).\)

**How to go from the general case to the special case**

Define

\[
v(x, t) = u\left(\frac{L}{\pi} x, t\right) .
\]

Then

\[
v_{xx} = \left(\frac{L}{\pi}\right)^2 u_{xx} ;
v(x, 0) = u\left(\frac{L}{\pi} x, 0\right) = f\left(\frac{L}{\pi} x\right) ;
v_t(x, 0) = u_t\left(\frac{L}{\pi} x, 0\right) = g\left(\frac{L}{\pi} x\right) ,
\]

and we get the problem

\[
\left(a \frac{\pi}{L}\right)^2 v_{xx} = v_{tt} \quad 0 < x < \pi \quad t > 0
\]

\[
v(0, t) = 0 \quad v(\pi, t) = 0 \quad t > 0
\]

\[
v(x, 0) = f\left(\frac{L}{\pi} x\right) ,
v_t(x, 0) = g\left(\frac{L}{\pi} x\right) ,
\]

**Important Solution**

(Special case \( L = \pi \))

The solution of the boundary value wave equation

\[
a^2 v_{xx} = v_{tt} \quad 0 < x < \pi \quad t > 0 ;
\]

\[
v(0, t) = 0 \quad v(\pi, t) = 0 \quad t > 0 ;
\]

\[
v(x, 0) = f(x) \quad v_t(x, 0) = g(x) \quad 0 < x < \pi .
\]
is
\[ v(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos(nat) + B_n \sin(nat) \right) \sin(nx) , \]
where the numbers \( A_n \) and \( B_n \) are given by the formulas
\[
A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx \\
B_n = \frac{2}{n\pi a} \int_{0}^{\pi} g(x) \sin nx \, dx.
\]
Doing the above reduction, we get

**Important Solution** (General Case)

The solution of the boundary value wave equation
\[
a^2 u_{xx} = u_{tt} , \quad 0 < x < L , \quad t > 0 ;
\]
\[
u(0, t) = 0 , \quad u(L, t) = 0 , \quad t > 0 ;
\]
\[
u(x, 0) = f(x) , \quad \nu_t(x, 0) = g(x) , \quad 0 < x < L .
\]
is
\[
u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos(n\pi a L t) + B_n \sin(n\pi a L t) \right) \sin(n\pi L x) ,
\]
where the numbers \( A_n \) and \( B_n \) are given by the formulas
\[
A_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx \\
B_n = \frac{2}{n\pi a} \int_{0}^{L} g(x) \sin \frac{n\pi x}{L} \, dx.
\]

**Problem 16.1** Solve the boundary value pde problem:
\[
u_{xx} = \nu_{tt} , \quad 0 < x < \pi , \quad t > 0
\]
\[
u(0, t) = 0 , \quad \nu(\pi, t) = 0 , \quad t > 0
\]
\[
u(x, 0) = x(\pi - x) , \quad \nu_t(x, 0) = 2x(\pi - x) , \quad 0 < x < \pi
\]

**Solution** Here \( a = 1 \) so:
\[
u(x, t) = \sum_{n=1}^{\infty} \left( A_n \cos(nt) + B_n \sin(nt) \right) \sin(nx) ,
\]
where

\[ A_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin nx \, dx \]

\[ B_n = \frac{2}{n\pi} \int_0^\pi 2x(\pi - x) \sin nx \, dx. \]

Using Maple, or integration tables,

\[ \int_0^\pi x(\pi - x) \sin nx \, dx = \frac{2(1 - (-1)^n)}{n^3} \]

So

\[ A_n = \frac{2}{\pi} \cdot \frac{2(1 - (-1)^n)}{n^3} = \frac{4(1 - (-1)^n)}{\pi n^3}, \]

\[ B_n = \frac{2}{n\pi} \cdot 2 \cdot \frac{2(1 - (-1)^n)}{n^3} = \frac{8(1 - (-1)^n)}{\pi n^4}. \]

Combining everything, we get:

\[ u(x, t) = \sum_{n=1}^{\infty} \left( \frac{4(1 - (-1)^n)}{\pi n^3} \cos(nt) + \frac{8(1 - (-1)^n)}{\pi n^4} \sin(nt) \right) \sin(nx). \]

This is the answer.

**Problem 16.2** Solve the boundary value problem

\[ u_{xx} = u_{tt}, \quad 0 < x < 4\pi, \quad t > 0; \]

\[ u(0, t) = 0, \quad u(4\pi, t) = 0, \quad t > 0; \]

\[ u(x, 0) = x(4\pi - x), \quad u_t(x, 0) = 2x(4\pi - x), \quad 0 < x < 4\pi. \]

by first solving a similar problem for the interval [0, \pi], and then going back.

**Solution.** We have to “squeeze” the interval [0, 4\pi] by a ratio of 4, so we define a new function

\[ v(x, t) = u(4x, t). \]

So \( v_{xx} = 4^2 u_{xx} \) and \( v(x, 0) = u(4x, 0) = 4x(4\pi - 4x) = 16x(\pi - x) \), \( v_t(4x, 0) = 2(4x)(4\pi - 4x) = 32x(\pi - x) \).

We have the new problem

\[ (\frac{1}{4})^2 v_{xx} = v_{tt}, \quad 0 < x < \pi, \quad t > 0; \]

\[ v(0, t) = 0, \quad v(\pi, t) = 0, \quad t > 0; \]

\[ v(x, 0) = 16x(\pi - x), \quad v_t(x, 0) = 32(\pi - x), \quad 0 < x < \pi. \]
Using the ready-made formula we have (here $a = \frac{1}{4}$)

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos(nt/4) + B_n \sin(nt/4)) \sin(nx),$$

where the numbers $A_n$ and $B_n$ are given by the formulas

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx,$$

$$B_n = \frac{2}{n\pi(1/4)} \int_{0}^{\pi} g(x) \sin nx \, dx.$$

We have

$$f(x) = 16x(\pi - x), \quad g(x) = 32x(\pi - x).$$

So

$$A_n = \frac{2}{\pi} \int_{0}^{\pi} 16x(\pi - x) \sin nx \, dx = \frac{32}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx,$$

$$B_n = \frac{256}{n\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx.$$

Using Maple, or integration tables,

$$\int_{0}^{\pi} x(\pi - x) \sin nx \, dx = \frac{2(1 - (-1)^n)}{n^3}.$$

So

$$A_n = \frac{32}{\pi} \int_{0}^{\pi} x(\pi - x) \sin nx \, dx = \frac{32}{\pi} \cdot \frac{2(1 - (-1)^n)}{n^3} = \frac{64(1 - (-1)^n)}{\pi n^3},$$

$$B_n = \frac{256}{n\pi} \cdot \frac{2(1 - (-1)^n)}{n^3} = \frac{512(1 - (-1)^n)}{\pi n^4}.$$

Combining, we get:

$$v(x, t) = \sum_{n=1}^{\infty} \left( \frac{64(1 - (-1)^n)}{\pi n^3} \cos(nt/4) + \frac{512(1 - (-1)^n)}{\pi n^4} \sin(nt/4) \right) \sin(nx).$$

Finally, we must go back to $u(x, t)$, using

$$u(x, t) = v(x/4, t).$$

So:

$$u(x, t) = \sum_{n=1}^{\infty} \left( \frac{64(1 - (-1)^n)}{\pi n^3} \cos((2k + 1)t/4) + \frac{512(1 - (-1)^n)}{\pi n^4} \sin((2k + 1)t/4) \right) \sin((2k + 1)x/4).$$

Since $(1) - (-1)^n$ is 0 when $n$ is even, and is 2 when $n$ is odd, we can write this even better as:

$$u(x, t) = \sum_{k=0}^{\infty} \left( \frac{128}{\pi(2k + 1)^3} \cos((2k + 1)t/4) + \frac{1024}{\pi(2k + 1)^4} \sin((2k + 1)t/4) \right) \sin((2k + 1)x/4).$$

Answer to 16.2:

$$u(x, t) = \sum_{k=0}^{\infty} \left( \frac{128}{\pi(2k + 1)^3} \cos((2k + 1)t/4) + \frac{1024}{\pi(2k + 1)^4} \sin((2k + 1)t/4) \right) \sin((2k + 1)x/4).$$