Important Definition

A linear second-order partial differential equation in two variables, for the dependent variable (alias function) \( u(x, y) \), depending on the independent variables, \( x, y \), is something of the form

\[
A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = G ,
\]

where \( A, B, C, D, E, F, G \) are either constants or functions of \( x \) and \( y \).

When \( G = 0 \) the pde is called homogeneous, otherwise it is nonhomogeneous.

A solution of a pde is any function that satisfies it (and it makes sense to differentiate it twice).

Note: There are lots of solutions to any given pde, so you need more information, besides the equation. These are given by boundary conditions (if one of the variables is time, then sometimes one uses the name initial conditions, for \( t = 0 \)).

Separation of Variables.

A function of two variables, \( u(x, y) \) is separable if it can be written as a product of two functions of one variable.

\[
u(x, y) = X(x)Y(y) .
\]

Problem 13.1: Find product solutions, if possible, to the partial differential equation

\[
\frac{\partial u}{\partial x} - 2 \frac{\partial u}{\partial y} = 0 .
\]

Solution: Write

\[
u(x, y) = X(x)Y(y) .
\]

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(X(x)Y(y)) = X'(x)Y(y) ,
\]

\[
\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(X(x)Y(y)) = X(x)Y'(y) .
\]

Going back to the equation, we have:

\[
X'(x)Y(y) - 2X(x)Y'(y) = 0 .
\]

So

\[
X'(x)Y(y) = 2X(x)Y'(y) .
\]
Dividing both sides by $X(x)Y(y)$, we get

$$\frac{X'(x)}{X(x)} = 2 \frac{Y'(y)}{Y(y)} = k .$$

The left side only depends on $x$. The right side only depends on $y$. In other words the left side is independent of $y$, the right side is independent of $x$, and since they are equal, both sides are independent of both variables ($x$ and $y$). This means that they are both equal to a constant. Let’s call this constant $k$. We have

$$\frac{X'(x)}{X(x)} = 2 \frac{Y'(y)}{Y(y)} = k .$$

So we have two ODE’s to solve

$$\frac{X'(x)}{X(x)} = k ,$$

and

$$2 \frac{Y'(y)}{Y(y)} = k .$$

Transposing

$$X' - kX = 0 , \quad Y' - (k/2)Y = 0 .$$

From Calc4,

$$X(x) = c_1 e^{kx} ,$$

$$Y(y) = c_2 e^{(k/2)y} ,$$

and finally we get

$$u(x, y) = X(x)Y(y) = c_1 c_2 e^{kx + 2ky} = Ce^{k(x+y/2)} .$$

($c_1, c_2$ are arbitrary constants), so we can combine them into $C = c_1 c_2$).

**Ans. to 13.1:** $u(x, y) = Ce^{k(x+y/2)}$, where $C$ and $k$ arbitrary constants.

**Problem 13.2:** Find product solutions, if possible, to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

**Solution:** Let $u = XY$, and we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = X''Y + X'Y' + XY''$$

So we have to solve

$$X''Y + X'Y' + XY'' = 0 .$$

There is no way to separate this, so this pde is not separable.

**Ans. to 13.2:** Not separable.
Important Definition:

A homogeneous partial differential equation

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = 0 \]

where \( A, B, C, D, E, F \) are real constants, is called

- **hyperbolic** if \( B^2 - 4AC > 0 \)
- **parabolic** if \( B^2 - 4AC = 0 \)
- **elliptic** if \( B^2 - 4AC < 0 \)

**Problem 13.3**: Classify the following pde

\[ 3 \frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} + 11 \frac{\partial u}{\partial x} \]

**Solution**: We first bring everything to the left, leaving 0 on the right:

\[ 3 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 4 \frac{\partial^2 u}{\partial y^2} - 11 \frac{\partial u}{\partial x} = 0 \]

In this problem \( A = 3, B = 1, C = -4 \). \( B^2 - 4AC = 1^2 - 4(3)(-4) = 1 + 48 = 49 \). Since this is positive, the pde is **hyperbolic**.

Ans. to 13.3: **hyperbolic**.

**Problem 13.4**: Classify the following pde

\[ 3 \frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 u}{\partial x \partial y} - 4 \frac{\partial^2 u}{\partial y^2} + 11 \frac{\partial u}{\partial x} \]

**Solution**: We first bring everything to the left, leaving 0 on the right:

\[ 3 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} - 11 \frac{\partial u}{\partial x} = 0 \]

In this problem \( A = 3, B = 1, C = 4 \). \( B^2 - 4AC = 1^2 - 4(3)(4) = 1 - 48 = -47 \). Since this is negative, the pde is **elliptic**.

Ans. to 13.4: **elliptic**.

**Very Important Principle: THE SUPERPOSITION PRINCIPLE**

If \( u_1, \ldots, u_r \) are solutions of a **homogeneous linear pde**, then any linear combination

\[ c_1 u_1 + c_2 u_2 + \ldots + c_r u_r \]

is also a solution.
is also a solution of the same pde.

Problem 3.5:

(i) Check that \( u_1(x, y) = \sin(x + y) \) is a solution of

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0
\]

(ii) Check that \( u_2(x, y) = \cos(x + y) \) is a solution of

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0
\]

(iii) Check that \( u_3(x, y) = e^{x+y} \) is a solution of

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0
\]

(iv) Prove that

\[
u_4(x, y) = 11 \sin(x + y) - 14 \cos(x + y) + 19e^{x+y},
\]

is also a solution.

Solution: (i),(ii),(iii): You do it!

(iv): \( u_4(x, y) \) is a linear combination of \( u_1, u_2, u_3 \) so by the superposition principle it is also a solution.