Dr. Z.'s Calc5 Lecture 13 Handout

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Important Definition

A **linear second-order partial differential equation** in **two variables**, for the **dependent variable** (alias **function**) $u(x, y)$,depending on the **independent variables**, x, y, is something of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad ,$$

where A, B, C, D, E, F, G are either **constants** or **functions** of x and y.

When $G = 0$ the **pde** is called **homogeneous**, otherwise it is **nonhomogeneous**.

A **solution** of a **pde** is any function that satisfies it (and it makes sense to differentiate it twice).

Note: There are **lots** of solutions to any given pde, so you need more information, besides the equation. These are given by **boundary conditions** (if one of the variables is time, then sometimes one uses the name **initial conditions**, for $t = 0$).

Separation of Variables

A function of **two** variables, $u(x, y)$ is **separable** if it can be written as a **product** of two functions of **one** variable.

$$u(x, y) = X(x)Y(y) \quad .$$

Problem 13.1: Find product solutions, if possible, to the partial differential equation

$$\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} = 0 \quad .$$

Solution: Write

$$u(x, y) = X(x)Y(y) \quad .$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(X(x)Y(y)) = X'(x)Y(y) \quad ,$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(X(x)Y(y)) = X(x)Y'(y) \quad .$$

Going back to the equation, we have:

$$X'(x)Y(y) - 2X(x)Y'(y) = 0 \quad .$$

So

$$X'(x)Y(y) = 2X(x)Y'(y) \quad .$$

Dividing both sides by $X(x)Y(y)$, we get

$$\frac{X'(x)}{X(x)} = 2\frac{Y'(y)}{Y(y)} \quad .$$

The left side only depends on x. The right side only depends on y. In other words the left side is **independent** of y, the right side is **independent** of x, and since they are equal, **both sides** are independent of **both variables** (x and y). This means that they are both equal to a *constant*. Let's call this constant k. We have

$$\frac{X'(x)}{X(x)} = 2\frac{Y'(y)}{Y(y)} = k \quad .$$

So we have two **ode**'s to solve

$$\frac{X'(x)}{X(x)} = k \quad .$$

and

$$2\frac{Y'(y)}{Y(y)} = k \quad .$$

Transposing

$$X' - kX = 0 \quad , \quad Y' - (k/2)Y = 0 \quad .$$

From Calc4,

$$X(x) = c_1 e^{kx} \quad ,$$

$$Y(y) = c_2 e^{(k/2)y} \quad ,$$

and finally we get

$$u(x,y) = X(x)Y(y) = c_1 c_2 e^{kx+(k/2)y} = Ce^{k(x+y/2)} \quad .$$

(c_1, c_2 are arbitrary constants), so we can combine them into $C = c_1 c_2$).

Ans. to 13.1: $u(x,y) = Ce^{k(x+y/2)}$, where C and k arbitrary constants.

Problem 13.2: Find product solutions, if possible, to the partial differential equation

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial^2 y} = 0 \quad .$$

Solution: Let $u = XY$, and we get

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial^2 y} = X''Y + X'Y' + XY''$$

So we have to solve

$$X''Y + X'Y' + XY'' = 0 \quad .$$

There is no way to **separate** this, so this pde is **not separable**.

Ans. to 13.2: Not separable.

Important Definition:

A homogeneous partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \quad ,$$

where A, B, C, D, E, F are real constants, is called

$$\textbf{hyperbolic} \quad if \quad B^2 - 4AC > 0$$

$$\textbf{parabolic} \quad if \quad B^2 - 4AC = 0$$

$$\textbf{elliptic} \quad if \quad B^2 - 4AC < 0$$

Problem 13.3: Classify the following pde

$$3\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} + 11\frac{\partial u}{\partial x} \quad .$$

Solution: We first bring everything to the left, leaving 0 on the right:

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 4\frac{\partial^2 u}{\partial y^2} - 11\frac{\partial u}{\partial x} = 0 \quad .$$

In this problem $A = 3, B = 1, C = -4$. $B^2 - 4AC = 1^2 - 4(3)(-4) = 1 + 48 = 49$. Since this is **positive**, the pde is **hyperbolic**.

Ans. to 13.3: **hyperbolic**.

Problem 13.4: Classify the following pde

$$3\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} - 4\frac{\partial^2 u}{\partial y^2} + 11\frac{\partial u}{\partial x} \quad .$$

Solution: We first bring everything to the left, leaving 0 on the right:

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - 11\frac{\partial u}{\partial x} = 0 \quad .$$

In this problem $A = 3, B = 1, C = 4$. $B^2 - 4AC = 1^2 - 4(3)(4) = 1 - 48 = -47$. Since this is **negative**, the pde is **elliptic**.

Ans. to 13.4: **elliptic**.

Very Important Principle: THE SUPERPOSITION PRINCIPLE

If $u_1, \ldots, u_r$ are solutions of a **homogeneous linear pde**, then **any** linear combination

$$c_1 u_1 + c_2 u_2 + \ldots + c_r u_r \quad ,$$

is also a solution of the same pde.

Problem 3.5:

(i) Check that $u_1(x,y) = \sin(x+y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

(ii) Check that $u_2(x,y) = \cos(x+y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

(iii) Check that $u_3(x,y) = e^{x+y}$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

(iv) Prove that

$$u_4(x,y) = 11\sin(x+y) - 14\cos(x+y) + 19e^{x+y} \quad ,$$

is also a solution.

Solution: (i),(ii),(iii): You do it!

(iv): $u_4(x,y)$ is a **linear combination** of u_1, u_2, u_3 so by the **superposition principle** it is also a solution.