Dr. Z.'s Calc5 Lecture 1 Handout: Definition of the Laplace Transform

By Doron Zeilberger

Theory:

The Definition of the Laplace Transform.

Input: A function \( f(t) \) defined on the non-negative real axis \([0, \infty)\).

Output: Another function, of \( s \), given by :

\[
F(s) = \int_0^\infty f(t)e^{-ts} \, dt .
\]

(Note, \( F(s) \) is also written \( \mathcal{L}\{f(t)\} \)).

What does it mean?

If you were promised a job from now \((t = 0)\) until eternity, with the income function \( f(t) \), paid continuously, being \( f(t) \), the total income would be

\[
\int_0^\infty f(t) \, dt ,
\]

and this is usually infinity. So you should be happy, you will earn infinite dollars! But what about inflation? If the continuous inflation rate is \( s \), so a dollar in \( t \) years would only be worth \( e^{-st} \) of today’s dollars, then your income after \( t \) years, would be

\[
e^{-st}f(t)
\]

in today’s dollars. The total income, from now to eternity would (usually) no longer be infinity but

\[
F(s) = \int_0^\infty f(t)e^{-ts} \, dt ,
\]

a certain finite amount that depends on the inflation rate \( s \).

For example, if you were promised an income function of \( f(t) = t \) and think that it is a great deal, and the inflation rate is 10\%, then your income would be

\[
F(1) = \int_0^\infty te^{-t/10} \, dt .
\]

Doing this integral (using integration by parts), we get

\[
F(1) = \int_0^\infty te^{-t/10} \, dt = (t)(-10e^{-t/10}) \bigg|_0^\infty - \int_0^\infty (-10e^{-t/10}) \, dt = 0 - 0 + 100 = 100 .
\]
(since \( \lim_{t \to \infty} te^{-t/10} = 0 \) and of course \( \lim_{t \to \infty} e^{-t/10} = 0 \))

So your effective total income would be only 100 dollars, in today’s dollars.

If the inflation rate is not so bad, %1 instead of %10 then if you do a similar calculation you would get 10000 dollars.

If you want to know a formula for any inflation rate, \( s \), then you do the above calculation for a symbolic \( s \) (only remembering that \( s \) must be positive, or else the inflation would be deflation and you would indeed be guaranteed an infinite income until you die at infinity). We have, instead of 0.1, \( s \) (but pretending that \( s \) is a number):

\[
F(s) = \int_0^\infty te^{-st} dt .
\]

Doing this integral (by integration by parts), we get

\[
F(s) = \int_0^\infty te^{-st} dt = \left[ \frac{t}{s} e^{-st} \right]_0^\infty - \int_0^\infty \left( -\frac{1}{s} e^{-st} \right) dt = \left( -\frac{1}{s^2} e^{-st} \right) _0^\infty = 0 - 0 - \frac{1}{s^2} = \frac{1}{s^2} .
\]

(since \( \lim_{t \to \infty} te^{-st} = 0 \) and of course \( \lim_{t \to \infty} e^{-st} = 0 \))

So your effective income would be only \( \frac{1}{s^2} \) dollars (in today’s dollars).

Note that the smaller your inflation rate, the larger would be your total income. For example, if \( s = 10^{-10} \) you would get \( 10^{20} \) dollars during your infinite life, a lot of money, but still not infinite!

**Problem 1.1:** Using the definition, find the Laplace transform \( \mathcal{L}\{f(t)\} \) (alias \( F(s) \)) of

\[
f(t) = 1 .
\]

**Solution:** By the definition

\[
\mathcal{L}\{f(t)\} = \int_0^\infty 1 \cdot e^{-st} dt = \int_0^\infty e^{-st} dt .
\]

Remember from Calc1 that, for any constant \( c \)

\[
\int e^{ct} dt = \frac{e^{ct}}{c} + C ,
\]

so

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^\infty = \frac{e^{-s \cdot \infty}}{-s} - \frac{e^{-s \cdot 0}}{-s} = \frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s} = \frac{1}{s} .
\]

**Ans. to 1.1:** The Laplace Transform of the constant function \( f(t) = 1 \) is the function (of \( s \)) \( F(s) = \frac{1}{s} \).
Problem 1.2: Using the definition, find the Laplace transform \( \mathcal{L}\{f(t)\} \) (alias \( F(s) \)) of 
\[ f(t) = e^t. \]

Solution: By the definition
\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^t \cdot e^{-st} \, dt = \int_0^\infty e^{(1-s)t} \, dt.
\]
Remember from Calc1 that, for any constant \( c \)
\[
\int e^{ct} \, dt = \frac{e^{ct}}{c},
\]
so (assuming, as we may, that \( s > 1 \))
\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{(1-s)t} \, dt = \frac{e^{(1-s)t}}{(1-s)} \bigg|_0^\infty = \frac{e^{(1-s)\infty} - e^{(1-s)0}}{1-s} = \frac{e^{-\infty} - e^{0}}{1-s} = \frac{1}{1-s} = \frac{1}{s-1}.
\]
Ans. to 1.2: The Laplace Transform of the function \( f(t) = e^t \) is the function (of \( s \)) \( F(s) = \frac{1}{s-1} \).

Comment: The above is valid only when \( s > 1 \).

Problem 1.3: Using the definition find the Laplace transform \( \mathcal{L}\{f(t)\} \) (alias \( F(s) \)) of
\[
 f(t) = \begin{cases} 
 1, & \text{if } 0 \leq t \leq 1; \\
 -1, & \text{if } t \geq 1.
\end{cases}
\]

Solution: By the definition
\[
\mathcal{L}\{f(t)\} = \int_0^\infty f(t) \cdot e^{-st} \, dt.
\]
Since \( f(t) \) is given by one formula in the interval \((0,1)\) and by another one in \((1,\infty)\), we have to break-up the integral and treat each piece separately.
\[
F(s) = \int_0^1 f(t)e^{-st} \, dt + \int_1^\infty f(t)e^{-st} \, dt
\]
\[
= \left[ \int_0^1 f(t)e^{-st} \, dt \right] + \left[ \int_1^\infty f(t)e^{-st} \, dt \right]
\]
\[
= \left[ \int_0^1 e^{-st} \, dt \right] + \left[ \int_1^\infty (-1) \cdot e^{-st} \, dt \right]
\]
\[
= \left[ \frac{e^{-st}}{-s} \right]_0^1 + \left[ \frac{e^{-st}}{-s} \right]_1^\infty
\]
\[
= \left( \frac{e^{-s} - e^{-0}}{-s} \right) - \left( \frac{e^{-s\infty} - e^{-s1}}{-s} \right)
\]
\[
= \left( \frac{e^{-s} - 1}{-s} \right) - \left( \frac{1 - e^{-s1}}{-s} \right)
\]
\[
= \frac{1}{s} - \frac{2e^{-s}}{s}.
\]
Ans. to 1.3: \( F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} \).
Using Tables of Laplace Transform

In real life, people (usually) don’t compute the Laplace Transform from scratch, they use tables. The most important items are:

\[
\begin{align*}
(a) \quad \mathcal{L}\{1\} &= \frac{1}{s} \\
(b) \quad \mathcal{L}\{t^k\} &= \frac{k!}{s^{k+1}} \quad (k = 1, 2, 3, \ldots), \\
(c) \quad \mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, \\
(d) \quad \mathcal{L}\{\sin kt\} &= \frac{k}{s^2 + k^2}, \\
(e) \quad \mathcal{L}\{\cos kt\} &= \frac{s}{s^2 + k^2}, \\
(f) \quad \mathcal{L}\{\sinh kt\} &= \frac{k}{s^2 - k^2}, \\
(g) \quad \mathcal{L}\{\cosh kt\} &= \frac{s}{s^2 - k^2}.
\end{align*}
\]

**Problem 1.4:** Using Tables, find \(\mathcal{L}\{f(t)\}\), if \(f(t) = 2(t+1)(t+4)\).

**Solution:** First use algebra to **expand**:

\[f(t) = 2(t+1)(t+4) = 2t^2 + 5t + 4 = 2t^2 + 10t + 8\quad .\]

Now use **linearity** and the tables

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{2t^2 + 10t + 8\} = \mathcal{L}\{2t^2\} + 10\mathcal{L}\{t\} + 8\mathcal{L}\{1\} = 2\frac{2!}{s^3} + 10\frac{1!}{s^2} + 8\frac{1}{s} = \frac{4}{s^3} + \frac{10}{s^2} + \frac{8}{s} .
\]

**Ans. to 1.4:** \(\mathcal{L}\{f(t)\} = \frac{4}{s^3} + \frac{10}{s^2} + \frac{8}{s} .
\]

**Note:** This is the best way to leave the answer, **Please** do not take common-denominator and simplify it further.

**Problem 1.5:** Using Tables, find \(\mathcal{L}\{f(t)\}\), if \(f(t) = 5 \sin 3t + (e^t + 1)^2\).

**Solution:** First use algebra to **expand**:

\[f(t) = 5 \sin 3t + e^{2t} + e^t + 1\quad .\]

By “linearity”

\[
\mathcal{L}\{f(t)\} = \mathcal{L}\{5 \sin 3t + e^{2t} + e^t + 1\} = 5\mathcal{L}\{\sin 3t\} + \mathcal{L}\{e^{2t}\} + \mathcal{L}\{e^t\} + \mathcal{L}\{1\} = 5\mathcal{L}\{\sin 3t\} + \mathcal{L}\{e^{2t}\} + 2\mathcal{L}\{e^t\} + \mathcal{L}\{1\} .
\]

Now use the tables:

\[
\mathcal{L}\{f(t)\} = 5 \cdot \frac{3}{s^2 + 9} + \frac{1}{s-2} + 2 \cdot \frac{1}{s-1} + \frac{1}{s} = \frac{15}{s^2 + 9} + \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s} .
\]

**Ans. to 1.5:** \(\mathcal{L}\{f(t)\} = \frac{15}{s^2 + 9} + \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s} .
\]