**Dr. Z.'s Calc5 Lecture 1 Handout: Definition of the Laplace Transform**

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**Theory:**

The **Definition of the Laplace Transform**.

**Input**: A function $f(t)$ defined on the non-negative real axis $[0, \infty)$.

**Output**: Another function, of $s$, given by :

$$F(s) = \int_0^\infty f(t) e^{-ts}\, dt \quad .$$

(Note, $F(s)$ is also written $\mathcal{L}\{f(t)\}$).

**What does it mean?**

If you were promised a job from now ($t = 0$) until eternity, with the income function , paid continuously, being $f(t)$, the total income would be

$$\int_0^\infty f(t) dt \quad ,$$

and this is usually infinity. So you should be happy, you will earn infinite dollars! But what about inflation? If the continuous inflation rate is $s$, so a dollar in $t$ years would only be worth $e^{-st}$ of today's dollars, then your income after $t$ years, would be

$$e^{-st} f(t)$$

in today's dollars. The total income, from now to eternity would (usually) no longer be infinity but

$$F(s) = \int_0^\infty f(t) e^{-ts}\, dt \quad ,$$

a certain *finite* amount that depends on the inflation rate $s$.

For example, if you were promised an income function of $f(t) = t$ and think that it is a great deal, and the inflation rate is %10, then your income would be

$$F(.1) = \int_0^\infty t e^{-t/10}\, dt \quad .$$

Doing this integral (using integration by parts), we get

$$F(.1) = \int_0^\infty t e^{-t/10}\, dt \quad = (t)(-10 e^{-t/10}) \Big|_0^\infty - \int_0^\infty (1)(-10 e^{-t/10})\, dt \quad =$$

$$-10 t e^{-t/10} \Big|_0^\infty - 100 e^{-t/10} \Big|_0^\infty = 0 - 0 + 0 + 100 = 100 \quad .$$

(since $\lim_{t\to\infty} te^{-t/10} = 0$ and of course $\lim_{t\to\infty} e^{-t/10} = 0$)

So your effective total income would be *only* 100 dollars, in today's dollars.

If the inflation rate is not so bad, %1 instead of %10 then if you do a similar calculation you would get 10000 dollars.

If you want to know a *formula* for *any* inflation rate, $s$, then you do the above calculation for a *symbolic* $s$ (only remembering that $s$ must be positive, or else the inflation would be deflation and you would indeed be guaranteed an infinite income until you die at infinity). We have, instead of 0.1, $s$ (but **pretending** that $s$ is a number):

$$F(s) = \int_0^\infty te^{-st}\,dt \quad .$$

Doing this integral ( by integration by parts), we get

$$F(s) = \int_0^\infty te^{-st}\,dt \quad = \frac{t}{s}e^{-st}\Big|_0^\infty - \int_0^\infty (1)(-(1/s)e^{-st})\,dt \quad =$$

$$\frac{t}{s}e^{-st}\Big|_0^\infty - \frac{1}{s^2}e^{-st}\Big|_0^\infty = 0 - 0 - (0 - \frac{1}{s^2}) = \frac{1}{s^2} \quad .$$

(since $\lim_{t\to\infty} te^{-st} = 0$ and of course $\lim_{t\to\infty} e^{-st} = 0$)

So your effective income would be *only* $\frac{1}{s^2}$ dollars (in today's dollars).

Note that the smaller your inflation rate, the larger would be your total income. For example, if $s = 10^{-10}$ you would get $10^{20}$ dollars during your infinite life, a lot of money, but still not infinite!

**Problem 1.1**: Using the **definition**, find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = 1 \quad .$$

**Solution**: By the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty 1 \cdot e^{-st}\,dt = \int_0^\infty e^{-st}\,dt \quad .$$

Remember from Calc1 that, for any constant $c$

$$\int e^{ct}\,dt = \frac{e^{ct}}{c} + C \quad ,$$

so

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\,dt \quad = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{e^{-s\cdot\infty}}{-s} - \frac{e^{-s\cdot 0}}{-s} = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = \frac{1}{s} \quad .$$

**Ans. to 1.1**: The Laplace Transform of the constant function $f(t) = 1$ is the function (of $s$) $F(s) = \frac{1}{s}$.

**Problem 1.2**: Using the **definition**, find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = e^t \quad .$$

**Solution**: By the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^t \cdot e^{-st}\, dt = \int_0^\infty e^{t-st}\, dt \quad = \int_0^\infty e^{(1-s)t}\, dt \quad .$$

Remember from Calc1 that, for any constant $c$

$$\int e^{ct}\, dt = \frac{e^{ct}}{c} \quad ,$$

so (assuming, as we may, that $s > 1$)

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{(1-s)t}\, dt \quad = \frac{e^{(1-s)t}}{(1-s)}\Big|_0^\infty = \frac{e^{(1-s)\infty}}{1-s} - \frac{e^{(1-s)\cdot 0}}{1-s} = \frac{e^{-\infty}}{1-s} - \frac{e^0}{1-s} = 0 - \frac{1}{1-s} = \frac{1}{s-1} \quad .$$

**Ans. to 1.2**: The Laplace Transform of the function $f(t) = e^t$ is the function (of $s$) $F(s) = \frac{1}{s-1}$.

**Comment**: The above is valid only when $s > 1$.

**Problem 1.3**: Using the **definition** find the Laplace transform $\mathcal{L}\{f(t)\}$ (alias $F(s)$) of

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1; \\ -1, & \text{if } t \ge 1. \end{cases}$$

**Solution** : By the definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) \cdot e^{-st}\, dt \quad .$$

Since $f(t)$ is given by one formula in the interval $(0,1)$ and by another one in $(1,\infty)$, we have to break-up the integral and treat each piece separately.

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt = \int_0^1 f(t) e^{-st}\, dt + \int_1^\infty f(t) e^{-st}\, dt$$

$$= \int_0^1 1 \cdot e^{-st}\, dt + \int_1^\infty (-1) \cdot e^{-st}\, dt = \int_0^1 e^{-st}\, dt + - \int_1^\infty e^{-st}\, dt$$

$$\frac{e^{-st}}{-s}\Big|_0^1 \quad - \quad \frac{e^{-st}}{-s}\Big|_1^\infty$$

$$= \left(\frac{e^{-s\cdot 1}}{-s} - \frac{e^{-s\cdot 0}}{-s}\right) - \left(\frac{e^{-s\cdot\infty}}{-s} - \frac{e^{-s\cdot 1}}{-s}\right) = -\frac{e^{-s\cdot 1}}{s} + \frac{1}{s} - 0 - \frac{e^{-s\cdot 1}}{s}$$

$$= \frac{1}{s} - \frac{2e^{-s}}{s} \quad .$$

**Ans. to 1.3**: $F(s) = \frac{1}{s} - \frac{2e^{-s}}{s}$.

**Using Tables of Laplace Transform**

In real life, people (usually) don't compute the Laplace Transform from stratch, they use tables. The most important items are:

$$(a) \quad \mathcal{L}\{1\} = \frac{1}{s}$$

$$(b) \quad \mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}} \quad (k = 1, 2, 3, ...),$$

$$(c) \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad ,$$

$$(d) \quad \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \quad ,$$

$$(e) \quad \mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \quad ,$$

$$(f) \quad \mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2} \quad ,$$

$$(g) \quad \mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2} \quad ,$$

**Problem 1.4**: Using Tables, find $\mathcal{L}\{f(t)\}$, if $f(t) = 2(t+1)(t+4)$.

**Solution**: First use algebra to **expand**:

$$f(t) = 2(t+1)(t+4) = 2(t^2+5t+4) = 2t^2+10t+8 \quad .$$

Now use **linearity** and the tables

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{2t^2+10t+8\} = \mathcal{L}\{2t^2\}+\mathcal{L}\{10t\}+\mathcal{L}\{8\} = 2\mathcal{L}\{t^2\}+10\mathcal{L}\{t\}+8\mathcal{L}\{1\} = 2\frac{2!}{s^3}+10\frac{1!}{s^2}+8\frac{1}{s} =$$

$$= \frac{4}{s^3} + \frac{10}{s^2} + \frac{8}{s} \quad .$$

**Ans. to 1.4**: $\mathcal{L}\{f(t)\} = \frac{4}{s^3} + \frac{10}{s^2} + \frac{8}{s}$.

**Note:** This is the best way to leave the answer, **Please** do not take common-denominator and simplify it further.

**Problem 1.5**: Using Tables, find $\mathcal{L}\{f(t)\}$, if $f(t) = 5\sin 3t + (e^t+1)^2$.

**Solution**: First use algebra to **expand**:

$$f(t) = 5\sin 3t + e^{2t} + 2e^t + 1 \quad .$$

By "linearity"

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{5\sin 3t+e^{2t}+2e^t+1\} = \mathcal{L}\{5\sin 3t\}+\mathcal{L}\{e^{2t}\}+\mathcal{L}\{2e^t\}+\mathcal{L}\{1\} = 5\mathcal{L}\{\sin 3t\}+\mathcal{L}\{e^{2t}\}+2\mathcal{L}\{e^t\}+\mathcal{L}\{1\} \quad .$$

**Now** use the tables:

$$\mathcal{L}\{f(t)\} = 5 \cdot \frac{3}{s^2+9} + \frac{1}{s-2} + 2 \cdot \frac{1}{s-1} + \frac{1}{s} = \frac{15}{s^2+9} + \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s} \quad .$$

**Ans. to 1. 5**: $\mathcal{L}\{f(t)\} = \frac{15}{s^2+9} + \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.