MATH 244 (1-3), Dr. Z. , Solutions to Pratice Exam I, for Dr. Z's Math 244(1-3), Fall 2016

1. (10 pts.) Find the general solution to the following differential equation

$$y''(t) + 100\,y(t) = 0 \quad .$$

Ans.: $y(t) = c_1 \sin 10t + c_2 \cos 10t$ (where c_1, c_2 are arbitrary constants)

The **characteristic equation** is $r^2 + 100 = 0$. Solving it we get $r^2 = -100$, so $r = \pm\sqrt{-100} = \pm\sqrt{100}\sqrt{-1} = 0 \pm 10i$.
So $\lambda = 0$ and $\mu = 10$. Plugging into the general formula

$$y(t) = e^{\lambda t}(c_1 \sin \mu t + c_2 \cos \mu t) \quad ,$$

we get

$$y(t) = e^{0 \cdot t}(c_1 \sin 10t + c_2 \cos 10t) = 1 \cdot (c_1 \sin 10t + c_2 \cos 10t) = c_1 \sin 10t + c_2 \cos 10t \quad .$$

Note: Some people added "steady oscillation". This is true, but I **never** asked what kind of oscillations it is. This time I was nice, and didn't take any points off, but next time, if I will ask you "Who is the president of the USA?" and you would answer "Mr. Obama is the president, and Mr. Biden is the vice-president' I will take points off. Please answer what has been asked! Not less, but also **not** more!

2. (10 pts.) Find the Wronskian, $W(f(t), g(t))$ of the following pair of functions:

$$f(t) = e^{3t} \quad , \quad g(t) = te^{3t} \quad .$$

Ans.: $W(f(t), g(t)) = e^{6t}$.

$$W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t) \quad .$$

Here $f(t) = e^{3t}$ so $f'(t) = 3e^{3t}$.
Also $g(t) = te^{3t}$. By the **product rule**, followed by the **chain rule**

$$g'(t) = t'e^{3t} + t(e^{3t})' = 1 \cdot e^{3t} + t(3e^{3t}) = e^{3t}(1 + 3t) \quad .$$

So we have

$$W(f(t), g(t)) = e^{3t} \cdot e^{3t}(1 + 3t) - 3e^{3t} \cdot te^{3t} = e^{6t}(1 + 3t) - 3te^{6t}$$

$$= (1 + 3t - 3t)e^{6t} = 1 \cdot e^{6t} = e^{6t} \quad .$$

Comment: Most people got it right, but some people forgot (or never knew) how to apply the product and chain rules. More depressingly, some people have trouble simplifying

$$e^{3t} \cdot e^{3t} \quad .$$

This is **basic algebra**. If you are having trouble, please review this! It is much more important than differential equations!

3. (10 pts.) Solve the initial value problem

$$y''(t) - 3\,y'(t) = 0 \quad , \quad y(0) = 2 \quad , \quad y'(0) = 3 \quad .$$

Ans.: $y(t) \ = 1 + e^{3t}$.

The **characteristic equation** is

$$r^2 - 3r = 0 \quad .$$

For this one it is much easier to factorize than to use the quadratic formula (that some people did and messed up!).

$$r(r-3) = 0 \quad .$$

Which is the same as

$$(r-0)(r-3) = 0 \quad .$$

So we have **two** distinct real roots $r_1 = 0$ and $r_2 = 3$. The general solution in this case is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad ,$$

so in our case it is

$$y(t) = c_1 e^{0 \cdot t} + c_2 e^{3 \cdot t} = c_1 + c_2 e^{3t} \quad .$$

Now it is time to find the constants by using the **initial conditions** $y(0) = 2$ and $y'(0) = 3$. For future reference

$$y'(t) = 3c_2 e^{3t} \quad ,$$

so

$$y(0) = c_1 + c_2 \quad , \quad y'(0) = 3c_2 \quad .$$

Taking advantage of the initial conditions we get the system of two linear equations with two unknowns, c_1, c_2:

$$c_1 + c_2 = 2 \quad , \quad 3c_2 = 3 \quad .$$

From the second we get that $c_2 = 3/3 = 1$. Plugging into the first equation, we get $c_1 + 1 = 2$ so $c_1 = 1$. Going back to the general solution $y(t) = c_1 + c_2 e^{3t}$, and substituting $c_1 = 1, c_2 = 1$ we get

$$y(t) = 1 + 1 \cdot e^{3t} = 1 + e^{3t} \quad .$$

4. (10 pts.) Use the **Euler method** to find an approximate value for $y(1.2)$ if $y(x)$ is the solution of the initial value problem differential equation

$$y' = x + y \quad , \quad y(1) = 0 \quad ,$$

using mesh-size $h = 0.1$.
Reminder: $x_n = x_0 + nh$, $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$.

Ans.: $y(1.2)$ is approximately equal to: 0.22 (or $\frac{11}{50}$).

Here $x_0 = 1$ (since the initial condition is $y(1) = 0$ and the argument of y is 1). Also $h = 0.1$, so

$$x_0 = 1 \quad , \quad x_1 = 1.1 \quad , \quad x_2 = 1.2 \quad .$$

Also $y_0 = 0$ (since the initial condition is $y(1) = 0$ and the right side is 0). In this problem

$$f(x, y) = x + y \quad .$$

Using $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, with $n = 1$ we have

$$y_1 = y_0 + (.1) \cdot f(x_0, y_0)) = 0 + 0.1 \cdot f(1, 0) = 0.1 \cdot (1 + 0) = 0.1 \quad .$$

Using $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, with $n = 2$ we have

$$y_2 = y_1 + (.1) \cdot f(x_1, y_1) = 0.1 + 0.1 \cdot f(1.1, 0.1) = 0.1 + 0.1 \cdot (1.1 + 0.1) = 0.1 + 0.1 \cdot (1.2) = 0.1 + 0.12 = 0.22 \quad .$$

This is the **answer**.

Comment: Most people got it right, but some people went 'over and above the call of duty' and went one more step with $n = 3$, and got an approximation to $y(1.3)$. They got only 5 points out of 10, and even this is charity. You have to answer what I asked for!

5. (10 pts.) For the following first-order differential equation, decide whether or not it is exact. If it is, solve it. Leave the answer in **implicit format**.

$$(3x^2 + y) + (x + 2y)\, y' = 0$$

Ans.: $x^3 + xy + y^2 = C$ (where C is an arbitrary constant) .

$$M = 3x^2 + y \quad , \quad N = x + 2y \quad .$$

$$M_y = 1 \quad , \quad N_x = 1 \quad ,$$

so $M_y = N_x$ and the diff.eq. is indeed **exact**.

$$F(x, y) = \int M(x, y)\, dx = \int (3x^2 + y)\, dx = x^3 + xy + \phi(y)$$

where $\phi(y)$ is a function of y **alone** yet TBD. Using $F_y = N$ we get

$$x + \phi'(y) = x + 2y \quad .$$

Thanks to algebra:

$$\phi'(y) = 2y \quad .$$

Integrating with respect to y:

$$\phi(y) = \int (2y)\, dy = y^2 \quad .$$

Going back to $F(x, y)$ we get

$$F(x, y) = x^3 + xy + y^2 \quad .$$

THIS IS NOT THE FINAL ANSWER, People who put this, got at most five out of the ten points. The Final answer is $F(x, y) = C$, where C is an arbitrary constant.

6. (10 pts.) For the following diff. eq. determine the critical (equilibrium) solutions and decide, for each such solution, whether it is asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y^2 - 3y \quad .$$

Ans.: $y = 0$, asymptoticlly stable ; $y = 3$, asymptoticlly unstable .

This is an **autonomous** diff.eq. Setting the right side equal to 0

$$y^2 - 3y = 0 \quad ,$$

and solving

$$y(y-3) = 0 \quad ,$$

we get **two** equilibritum solutions $y = 0$ and $y = 3$. Let's investigate them each.
For $y = -.1$, $y' = (-0.1)(-0.1-3)$ is **positive** so it tends to go up. For $y = .1$, $y' = (0.1)(0.1-3)$ is **negative** so it tends to go down. So in either case it tends to go **towards** $y = 0$ and it is **stable**.
For $y = 2.9$, $y' = (2.9)(2.9-3)$ is **negative** so it tends to go down For $y = 3.1$, $y' = (3.1)(3.1-3)$ is **positive** so it tends to go up. So in either case it tends to **away** from $y = 3$ and it is **unstable**.

7. (10 pts.) Find the maximal open interval for which the following first-order diff.eq. inital value problem is guaranteed to have a unique solution. **Explain!**

$$(t-3)(t+4)\,y'(t) + e^t\,y(t) = t^2 \quad , \quad y(-1) = 10 \quad .$$

Ans.: $-4 < t < 3$.

Since this is an **initial value problem** we are only interested in what is going on near $t = -1$. Dividing by the coefficient of $y'(t)$ we get inital value problem is guaranteed to have a unique solution. **Explain!**

$$y'(t) + \frac{e^t}{(t-3)(t+4)}\,y(t) = \frac{t^2}{(t-3)(t+4)} \quad , \quad y(-1) = 10 \quad .$$

The coefficient of $y(t)$ and the right side blow-up at $t = -4$ and at $t = 3$ (since then we have division by 0), but as long as you stay in the interval $-4 < t < 3$ things are nice and calm. So the maximal interval is indeed $-4 < t < 3$.

Comment: Some people gave as the answer the three intervals

$$-\infty < t < -4 \quad , \quad -4 < t < 3 \quad , \quad 3 < t < \infty \quad , \quad .$$

This is the right answer to a **different** question: "Find the maximal open intervals (in plural!) for which there are unique solutions to the diff. eq. $(t-3)(t+4)\,y'(t)+e^t\,y(t) = t^2$" (where no inital condition is given). Don't confuse the two kinds of problems!
I was being nice this time and gave 6 out of 10 points for the peole who gave this anwser.
I won't be as nice next time!

8. (10 pts.) Find an equation of the curve that passes through the point $(1, 2)$ and whose slope at (x, y) is x/y.

Ans.: $y^2 - x^2 = 3$ or $y = \sqrt{x^2+3}$.

Since **slope is derivative**, we have to solve the diff.eq.

$$\frac{dy}{dx} = \frac{x}{y} \quad .$$

Since the curve should pass through the point $(1, 2)$ we need to really solve the IVP

$$\frac{dy}{dx} = \frac{x}{y} \quad , \quad y(1) = 2 \quad .$$

The diff.eq. is handled via the method of **separation of variables**.

$$\int y\, dy = \int x\, dx \quad .$$

Doing the integration:

$$\frac{y^2}{2} = \frac{x^2}{2} + C \quad .$$

Multiplying by 2:

$$y^2 = x^2 + C \quad .$$

(since $2C = C$). Now we plug-in $x = 1$ $y = 2$ and find out what C is:

$$2^2 = 1^2 + C \quad .$$

So

$$C = 4 - 1 = 3 \quad .$$

Going back to the general solution we get

$$y^2 = x^2 + 3 \quad ,$$

and a little bit nicer, in **implicit** format $y^2 - x^2 = 3$ (BTW this is a hyperbola). This is acceptable as a final answer, since this is a **curve** (with two components, but that's OK). But some people love **explicit** answers and took the sqaure-root, getting

$$y = \pm\sqrt{x^2+3} \quad .$$

That's very nice of them. Indeed they are good students, and remember the solution of $x^2 = a$ is $\pm\sqrt{a}$. **But** the curve $y = -\sqrt{x^2+3}$ does not pass through the point $(1, 2)$ so it must be discarded. So those people who insist on explicit answers, shoud have had $y = \sqrt{x^2+3}$ without the $\pm$.

The irony is that people who are not-so-good-students, and forgot about the $\pm$ would have gotten it completely right.

I only took two points off for having $\pm$.

9. (10 pts.) Solve the initial value problem

$$y'(t) - 3\,y(t) = e^{2t} \quad , \qquad y(0) = 1.$$

Ans.: $y(t) = 2e^{3t} - e^{2t}$.

We use the method of **integrating factor**.

$$p(t) = -3 \quad ,$$

so

$$I(t) = e^{\int p(t)\,dt} = e^{\int -3\,dt} = e^{-3t} \quad .$$

Multiplying the diff. eq. by $I(t) = e^{-3t}$ we get:

$$e^{-3t}y'(t) - 3e^{-3t}\,y(t) = e^{2t} \cdot e^{-3t} = e^{-t} \quad .$$

The left side is **guranteed** to be $(I(t)y(t))'$ (but it is a good idea to check), so

$$(e^{-3t}y(t))' = e^{-t} \quad .$$

Integrating:

$$e^{-3t}y(t) = \int e^{-t}\,dt = -e^{-t} + C \quad .$$

Warning: Don't forget the $+C$!
Dividing by e^{-3t} we get

$$y(t) = \frac{-e^{-t} + C}{e^{-3t}} = -e^{2t} + Ce^{3t} \quad .$$

Now it is time to use the **initial condition**, $y(0) = 1$.

$$y(0) = -e^{2\cdot 0} + Ce^{3\cdot 0} = -1 + C \quad .$$

But $y(0) = 1$ so

$$1 = -1 + C \quad .$$

Solving for C, we get $C = 2$. Finally going back to the general solution $y(t) = -e^{2t} + Ce^{3t}$ and pugging-in $C = 2$ we get the final answer

$$y(t) = -e^{2t} + 2e^{3t} \quad .$$

10. (10 pts.) Decide whether $y(t) = te^{2t}$ is a solution of the initial value differential equation

$$y''(t) - 4\,y'(t) + 4\,y(t) \;=\; 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad .$$

Explain everything!

Comment: This is a problem from Lecture 1. It could also be done by actually solving it, using the method of Lecture 11 (that **was** not part of this exam). People who did it correctly got full credit, but they were really supposed to do it the Assignment 1 way, since it says 'decide' not solve!

$$y(t) = te^{2t}$$

By the product and chain rules:

$$y'(t) = (te^{2t})' = t'e^{2t} + t(e^{2t})' = 1 \cdot e^{2t} + t(2e^{2t}) = (1+2t)e^{2t} \quad .$$

$$y''(t) = ((1+2t)e^{2t})' = (1+2t)'e^{2t} + (1+2t)(e^{2t})' = 2e^{2t} + (1+2t)(2e^{2t}) = (4+4t)e^{2t} \quad .$$

Going to the diff.eq. and simplifying, we get

$$y''(t) - 4\,y'(t) + 4\,y(t) \;=\; (4+4t)e^{2t} - 4\cdot(1+2t)e^{2t} + 4\cdot te^{2t} = (4+4t-4-8t+4t)e^{2t} = 0\cdot e^{2t} = 0 \quad .$$

Yea! we got something right, so the proposed function is indeed a solution of the diff.eq. We still have to worry about the **initial conditions**.

$$y(0) = 0 \cdot e^{2\cdot 0} = 0$$

$$y'(0) = (1 + 2\cdot 0)e^{2\cdot 0} = 1 \quad .$$

So the initial conditions are also OK!