Solutions to Attendance Quiz # 21 for Dr. Z.'s Calc4 for Lecture 21

1. Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad .$$

Sol. to 1: We first form the **characteristic matrix**

$$\begin{pmatrix} 4-r & -8 \\ 2 & -4-r \end{pmatrix} \quad .$$

The **determinant** is

$$(4-r)(-4-r) - (-8)(2) = (r-4)(r+4) + 16 = r^2 - 16 + 16 = r^2 \quad .$$

Hence the **characteristic equation** is

$$r^2 = 0 \quad .$$

This has **one** *repeated root*, $r = 0$. So we only have **one** eigenvalue, $r = 0$.

It is time to find an eigenvector (or, if lucky two independent eignevectors, but this usually does not happen).

When $r = 0$ the characteristic matrix is

$$\begin{pmatrix} 4-0 & -8 \\ 2 & -4-0 \end{pmatrix} = \begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} \quad .$$

We are looking for an **eigenvector** $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, such that

$$\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad .$$

Spelling it out, we have to solve the system

$$4c_1 - 8c_2 = 0 \quad ,$$

$$2c_1 - 4c_2 = 0 \quad .$$

The first equation is a multiple of the first, so it is enough to consider the second, giving

$$c_1 = 2c_2 \quad .$$

Going back to the general **eigenvector** $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ we have

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2c_2 \\ c_1 \end{pmatrix} = c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad .$$

SO we found **infinitely many** eigenvectors, but all we need is **one**, so we plug-in *anything* for c_2 (EXCEPT FOR ZERO!). Since $c_2 = 1$ is the simplest number (after 0), we take $c_2 = 1$ and get that an eigenvector corresponding to $r = 0$ is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and we got the **first fundamental solution**

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{0 \cdot t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad .$$

Since we only got one (independent) eigenvector, now it it is time to find a second fundamental solution.

By general theory

$$\mathbf{x}_2(t) = t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathbf{u} e^{0 \cdot t} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mathbf{u} \quad ,$$

where the vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is **to be determined**. It is a solution of $(\mathbf{P} - r\mathbf{I}))\mathbf{u} = \mathbf{v}$, where $\mathbf{v}$ is the eigenvector above. In this problem $r = 0$, so we have to solve

$$\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad .$$

Spelling it out:

$$4u_1 - 8u_2 = 2 \quad ,$$

$$2u_1 - 4u_2 = 1 \quad ,$$

The first equation is double the second, so we only need to consider the second equation, getting

$$u_1 = 2u_2 + \frac{1}{2} \quad .$$

Going back to the general $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ we have

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2u_2 + \frac{1}{2} \\ u_2 \end{pmatrix} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = u_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad .$$

Now we can pick anything (including zero!) for u_2, so let's put $u_2 = 0$ and get that

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

Going back to the $\mathbf{x}_2(t)$ above:

$$\mathbf{x}_2(t) = t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad .$$

Now is the time to write down the **general solution**

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) \quad ,$$

so

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right) \quad ,$$

where c_1, c_2 are **arbitrary constants**.

Now it is time to incorporate the **initial condition** $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Plugging-in $t = 0$ into the general solution yields

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \left(0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} 2c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_1 + \frac{1}{2}c_2 \\ c_1 \end{pmatrix} \quad .$$

So we have

$$\begin{pmatrix} 2c_1 + \frac{1}{2}c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad .$$

Spelling it out:

$$2c_1 + \frac{1}{2}c_2 = -1 \quad ,$$

$$c_1 = 1 \quad .$$

So $c_1 = 1$, and $2 \cdot 1 + \frac{1}{2}c_2 = -1$, so $\frac{1}{2}c_2 = -3$ and $c_2 = -6$. We got that

$$c_1 = 1 \quad , \quad c_2 = -6 \quad .$$

Going back to the general solution, and substituting the newly found c_1, c_2, we get

$$\mathbf{x}(t) = (1) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-6) \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 6 \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right) \quad .$$

This is **correct** but not simplified! To simplify, we do vector algebra:

$$\mathbf{x}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 12t \\ 6t \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 - 12t \\ 1 - 6t \end{pmatrix} \quad .$$

At long last, we got

Ans. to 1: $\mathbf{x}(t) = \begin{pmatrix} -12t - 1 \\ -6t + 1 \end{pmatrix}$.

Checking Let's check out answer. The easy part is the initial condition $\mathbf{x}(0) = \begin{pmatrix} -12 \cdot 0 - 1 \\ -6 \cdot 0 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Yea!

Now, the diff.eq. part. The left side is

$$\mathbf{x}'(t) = \begin{pmatrix} -12 \\ -6 \end{pmatrix} \quad .$$

The right side is

$$\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} -12t - 1 \\ -6t + 1 \end{pmatrix} = \begin{pmatrix} 4(-12t - 1) - 8(-6t + 1) \\ 2(-12t - 1) - 4(-6t + 1) \end{pmatrix}$$

$$\begin{pmatrix} -48t - 4 + 48t - 8) \\ -24t - 2 + 24t - 4 \end{pmatrix} = \begin{pmatrix} -12 \\ -6 \end{pmatrix} \quad .$$

YEA! we did not mess-up! The left side equals the right side, and we did it correctly!