Solutions to Attendance Quiz # 18 for Dr. Z.'s Calc4

1. Find the eigenvalues, and corresponding eigenvectors of the following matrix.

$$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$$

1. The characteristic equation is

$$\det \begin{pmatrix} -5-\lambda & 2 \\ -6 & 2-\lambda \end{pmatrix} = 0 \quad .$$

So

$$(-5-\lambda)(2-\lambda) - (2)(-6) = \lambda^2 + 3\lambda + 2 = 0$$

Factorizing

$$(\lambda+1)(\lambda+2) = 0 \quad ,$$

so the *eigenvalues* are $\lambda = -2$ and $\lambda = -1$.

Now it is time to find the eigenvectors.

For $\lambda = -2$ we solve

$$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad .$$

Spelling it out

$$-5x_1 + 2x_2 = -2x_1$$

$$-6x_1 + 2x_2 = -2x_2 \quad .$$

From the first equation we get $3x_1 = 2x_2$, so $x_1 = \frac{2}{3}x_2$. Pluging into the second equation gives us $0 = 0$ and that's good! So an eigenvector for $\lambda = -2$ is

$$\begin{pmatrix} \frac{2}{3}x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} \quad .$$

We can choose $x_2 = 1$, but we can pick **any** non-zero value, and $x_2 = 3$ is nicer (to clear fractions), so the best looking eigenvector for $\lambda = -2$ is

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad .$$

For $\lambda = -1$ we solve

$$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad .$$

Spelling it out

$$-5x_1 + 2x_2 = -x_1$$

$$-6x_1 + 2x_2 = -2x_2 \quad .$$

From the first equation we get $4x_1 = 2x_2$, so $x_2 = 2x_1$. Pluging into the second equation gives us $0 = 0$ and that's good! So an eigenvector for $\lambda = -1$ is

$$\begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad .$$

We can choose $x_1 = 1$. So an eigenvector for $\lambda = -1$ is

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad .$$

Ans. to 1: There are two eigenvalues, $\lambda = -2$ and $\lambda = -1$.

An eigenvector corresponding to $\lambda = -2$ is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. (and any **non-zero** multiple of it is also an eigenvector, of course)

An eigenvector corresponding to $\lambda = -1$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (and any **non-zero** multiple of it is also an eigenvector, of course)

2. Write the follwing system in matrix notation

$$x_1'(t) = -t\, x_3(t) - 2t\, x_2(t) + \frac{1}{t+1}$$

$$x_2'(t) = -x_3(t) - 3\, x_1(t) + \cos^2 t$$

$$x_3'(t) = -9\, x_2(t) + \tan t\, x_1(t) + 3t^2 x_3(t)$$

Sol. to 2: **First** we have to write $x_1(t), x_2(t), x_3(t)$ **in order**, and stick 0 in front of those that are missing

$$x_1'(t) = 0\, x_1(t) - 2t\, x_2(t) - t\, x_3(t) + \frac{1}{t+1}$$

$$x_2'(t) = -3\, x_1(t) + 0\, x_2(t) - x_3(t) + \cos^2 t$$

$$x_3'(t) = \tan t\, x_1(t) - 9\, x_2(t) + 3t^2 x_3(t)$$

Finally:

Ans. to 2:

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & -2t & -t \\ -3 & 0 & -1 \\ \tan t & -9 & 3t^2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} \frac{1}{t+1} \\ \cos^2 t \\ 0 \end{pmatrix} \quad .$$