**Solutions to Attendance Quiz # 15 for Dr. Z.'s Calc4 for Lecture 15**

**1.** Solve the initial value problem

$$y^{(4)}(t) - 5y''(t) + 4y(t) = 0 \quad ; \quad y(0) = 4 \quad , \quad y'(0) = 0 \quad , \quad y''(0) = 2 \quad , \quad y'''(0) = 8 \quad .$$

**Sol. to 1**: We first find the **General Solution**. The **characteristic equation** is

$$r^4 - 5r^2 + 4 = 0 \quad .$$

Factorizing (treating $r^2$ as $a$)

$$(r^2 - 1)(r^2 - 4) = 0 \quad .$$

Factorizing as much as possible:

$$(r + 1)(r - 1)(r + 2)(r - 2) = 0 \quad .$$

So we have **four** distinct real roots $-2, -1, 1, 2$, and the General Solution is

$$y(t) = c_1\, e^{-2t} + c_2\, e^{-t} + c_3\, e^{t} + c_4 e^{2t} \quad .$$

Where $c_1, c_2, c_3, c_4$ are arbitrary constants.

**Note:** Most people got this part correctly.

Now comes the 'fun' part. **First**, we need $y'(t), y''(t), y'''(t)$.

$$y'(t) = -2c_1\, e^{-2t} - c_2\, e^{-t} + c_3\, e^{t} + 2c_4 e^{2t} \quad .$$

$$y''(t) = 4c_1\, e^{-2t} + c_2\, e^{-t} + c_3\, e^{t} + 4c_4 e^{2t} \quad .$$

$$y'''(t) = -8c_1\, e^{-2t} - c_2\, e^{-t} + c_3\, e^{t} + 8c_4 e^{2t} \quad .$$

Plug-in $t = 0$:

$$y(0) = c_1 + c_2 + c_3 + c_4 \quad .$$

$$y'(0) = -2c_1 - c_2 + c_3 + 2c_4 \quad .$$

$$y''(0) = 4c_1 + c_2 + c_3 + 4c_4 \quad .$$

$$y'''(0) = -8c_1 - c_2 + c_3 + 8c_4 \quad .$$

Now we use the **initial conditions**, to get a *system* of four linear equations with four unknowns $c_1, c_2, c_3, c_4$

$$c_1 + c_2 + c_3 + c_4 = 4 \quad ,$$

$$-2c_1 - c_2 + c_3 + 2c_4 = 0 \quad ,$$

$$4c_1 + c_2 + c_3 + 4c_4 = 2 \quad .$$

$$-8c_1 - c_2 + c_3 + 8c_4 = 8 \quad .$$

Now comes the **really** fun part!

One way is to use *Gaussian elimination* using row-operations, but we are not supposed to know that. So we will use **back substitution**.

From the first equation we get

$$c_4 = 4 - c_1 - c_2 - c_3 \quad ,$$

Let's plug this into the last three equations.

$$-2c_1 - c_2 + c_3 + 2(4 - c_1 - c_2 - c_3) = 0 \quad ,$$

$$4c_1 + c_2 + c_3 + 4(4 - c_1 - c_2 - c_3) = 2 \quad .$$

$$-8c_1 - c_2 + c_3 + 8(4 - c_1 - c_2 - c_3) = 8 \quad .$$

Open-up parentheses!

$$-2c_1 - c_2 + c_3 + 8 - 2c_1 - 2c_2 - 2c_3 = 0 \quad ,$$

$$4c_1 + c_2 + c_3 + 16 - 4c_1 - 4c_2 - 4c_3 = 2 \quad .$$

$$-8c_1 - c_2 + c_3 + 32 - 8c_1 - 8c_2 - 8c_3 = 8 \quad .$$

Clean-up!

$$-4c_1 - 3c_2 - c_3 = -8 \quad ,$$

$$-3c_2 - 3c_3 = -14 \quad .$$

$$-16c_1 - 9c_2 - 7c_3 = -24 \quad .$$

Now we have to solve a system of three equations with three unknowns, $c_1, c_2, c_3$. Since the second equation does not have $c_1$, it is most efficient to take this as the 'pivot'. we get

$$-3c_2 = -14 + 3c_3 \quad ,$$

Hence

$$c_2 = \frac{14}{3} - c_3 \quad .$$

Now we plug this into the first and third equations:

$$-4c_1 - 3(\frac{14}{3} - c_3) - c_3 = -8 \quad ,$$

$$-16c_1 - 9(\frac{14}{3} - c_3) - 7c_3 = -24 \quad .$$

Open-up parentheses:

$$-4c_1 - 14 + 3c_3 - c_3 = -8 \quad ,$$

$$-16c_1 - 42 + 9c_3 - 7c_3 = -24 \quad .$$

Clean-up!

$$-4c_1 + 2c_3 = 6 \quad ,$$

$$-16c_1 + 2c_3 = 18 \quad .$$

**Now** we only have to solve a system of two equations and two unknowns, $c_1, c_3$.

From the first equation

$$2c_3 = 6 + 4c_1 \quad ,$$

divide by 2:

$$c_3 = 3 + 2c_1 \quad ,$$

Substitute into the second equation

$$-16c_1 + 2(3 + 2c_1) = 18 \quad .$$

Open-up parentheses (aka as *expand*):

$$-16c_1 + 6 + 4c_1 = 18 \quad .$$

Clean-up:

$$-12c_1 = 12$$

Divide by $-12$, we get

$$c_1 = -1 \quad .$$

Yea, we found **one** of these four guys. Now it is time to **backtrack**, by doing **back-substitution**.

$$c_3 = 3 + 2c_1 = 3 + 2(-1) = 1 \quad ,$$

So $c_3 = 1$.

**Two more to go!**

$$c_2 = \frac{14}{3} - c_3 = \frac{14}{3} - 1 = \frac{11}{3}$$

And *last but not least!*

$$c_4 = 4 - c_1 - c_2 - c_3 = 4 - (-1) - \frac{11}{3} - 1 = 4 - \frac{11}{3} = \frac{1}{3} \quad .$$

Summarizing: $c_1 = -1$, $c_2 = \frac{11}{3}$, $c_3 = 1$, $c_4 = \frac{1}{3}$.

**FINALLY**, we go back to the General Solution, and substitute these values.

**Ans. to 1**:

$$y(t) = -e^{-2t} + \frac{11}{3}\, e^{-t} + e^{t} + \frac{1}{3} e^{2t} \quad .$$

**Comments**: **1**. Most people tackled it the right way, **but** because it was so long, either didn't even try, or tried, but sooner or later messed up. Nevertheless, **a few poeple got the right answer** (congratulations!).

**2.** Such a complicated problem is not likely to show up on an exam. In fact, I am almost sure, that I 'cooked' a problem with a much nicer answer, but I entered the initial conditions differently, and I changed a relatively (computationally) easy problem to a nightmare. Sorry. Then again, it is good practice! In real life problems are usually complicated, and no artificially 'nice' answers.