**MATH 244 (1-3), Dr. Z. , SOLUTIONS to Exam II, Thurs. Dec. 1, 2016, 8:40-10:00am, SEC 117**

**Disclaimer:** Not responsible for any errors. The first finder of any error will get one dollar.

**GENERAL COMMENT**: This exam was harder than I intended. As a consequence many good students ran out of time. When I graded it, I was very generous with partial credit for people who were on the right track, but did not finish it, and also for people who just described in words what they would do if they had time. To make up for it, I lowered the "bar" for each grade by 10 points (for example, to get an A you now need 450 out of 500 instead of 460). (See the webpage

`http://www.math.rutgers.edu/ zeilberg/policy244_16.html` ).

The class average for this exam was 66 (out of 100), and when I taught it three years ago (with a much easier exam) the average was 76, so this "lowering the bar" makes up for this exam being too hard (for 80 minutes).

**Reminders about Stability of Critical Points**:

Two real and distinct eigenvalues **BOTH** positive, ($r_1 > r_2 > 0$): Node ; Unstable; Two real and distinct eigenvalues **BOTH** negative, ($r_1 < r_2 < 0$):Node ; Asymptotically Stable Two real and distinct eigenvalues of **OPPOSITE SIGN** ($r_2 < 0 < r_1$):Saddle Point ; Unstable

Repeated eigenvalue that is positive ($r_1 = r_2 > 0$):Proper or Improper Node ; Unstable (Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

Repeated eigenvalue that is negative ($r_1 = r_2 < 0$): Proper or Improper Node ; Asymptotically Stable (Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

Complex eigenvectors ( $r_1, r_2 = \lambda \pm i\mu$) with positive real part (i.e. $\lambda > 0$): Spiral point ; Unstable Complex eigenvectors ( $r_1, r_2 = \lambda \pm i\mu$) with negative real part, (i.e. $\lambda < 0$):Spiral point ; Asymptotically Stable

Complex eigenvectors ( $r_1, r_2 = \lambda \pm i\mu$) with zero real part, (i.e. $\lambda = 0$):Center ; Stable

**Reminder about the method of variation of parameters**: If the functions $p(t), q(t), g(t)$ are continuous on an open interval $I$, and if $y_1(t)$ and $y_2(t)$ are independent solutions of the **homogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0 \quad ,$$

then a particular solution of the **inhomogeneous** diff.eq.

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t) \quad ,$$

is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where $u_1(t), u_2(t)$ are two functions whose derivatives satisfy the system of two equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \quad ,$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t) \quad ,$$

**Reminder about reduction of order**: If $y_1(t)$ is a solution of the diff.eq. $y''(t) + p(t)y'(t) + q(t)y(t) = 0$, then to get, another, independent solution, $y_2(t)$, you write $y_2(t) = y_1(t)v(t)$, and solve the diff.eq, $y_1(t)v''(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0 \quad .$

**1.** (10 pts.) Solve the initial value problem

$$y'''(t) \; - \; y''(t) \; + \; y'(t) \; - \; y(t) = 2e^t \quad , \quad y(0) = 0 \quad , \quad y'(0) = 2 \quad , \quad y''(0) = 2 \quad .$$

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**Ans.**: $y(t) = \sin t + te^t \quad .$

**Type: Function of $t$.**

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We first must find the **General Solution** of the **Homogeneuos version** of the diff. eq., where the **right side** is made **zero** (and forget about the initial conditions until much later)

$$y'''(t) \; - \; y''(t) \; + \; y'(t) \; - \; y(t) = 0 \quad .$$

The **characteristic equation** obtained by replacing $y'''(t)$ by $r^3$, replacing $y''(t)$ by $r^2$, replacing $y'(t)$ by $r$, and $y(t)$ by 1 is:

$$r^3 - r^2 + r - 1 = 0 \quad .$$

We have to solve this **algebraic equation**. Factoring (by inspection it is easy to see that $r = 1$ is a root, and then it is easy to find the other factor).

$$(r-1)(r^2+1) = 0 \quad .$$

So the roots are $r = 1$ and $r = \pm\sqrt{-1} = 0 \pm 1 \cdot i$. Hence the **General Solution of the Homogeneous version** is

$$y(t) = c_1 \, e^t \; + \; c_2 \cos t \; + \; c_3 \sin t \quad .$$

Next, it is time to find a **Particular Solution**. The **default template** is

$$y(t) = Ae^t \quad ,$$

alas, it **conflicts** with one of the components of the above general solution (you are also welcome to try it out and get the nonsense $0 = 2$ indicating that it is no good).

So the **next thing** is to multiply the previous try by $t$, getting that the **updated template** is

$$y(t) = Ate^t \quad .$$

Now there is no more conflict, and we have to find the derivatives all the way to the third.

$$y'(t) = A(t'e^t + t(e^t)') = A(e^t + te^t) = A(t+1)e^t \quad ,$$

$$y''(t) = A((t+1)'e^t + (t+1)(e^t)') = A(e^t + (t+1)e^t) = A(t+2)e^t \quad ,$$

$$y'''(t) = A((t+2)'e^t + (t+2)(e^t)') = A(e^t + (t+2)e^t) = A(t+3)e^t \quad .$$

Plugging these all in the diff.eq. we get

$$A(t+3)e^t \; - \; A(t+2)e^t \; + \; A(t+1)e^t \; - \; Ate^t = 2e^t$$

Algebra:

$$Ae^t(t+3-(t+2)+(t+1)-t) = 2e^t \quad .$$

More algebra:

$$Ae^t(t+3-t-2+t+1-t) = 2e^t \quad ,$$

giving

$$Ae^t \cdot 2 = 2e^t \quad .$$

Dividing by $e^t$ we get

$$A = 1 \quad .$$

Going back to the template, we found out that a **particular solution**, $y_P(t)$ is

$$y_P(t) = te^t \quad .$$

Now we have the **General Solution** of our (inhomog.) diff. eq.

$$y(t) = c_1\, e^t \; + \; c_2 \cos t \; + \; c_3 \sin t + te^t \quad .$$

Now it is time to figure out the actual values of $c_1, c_2, c_3$ that would satisfy the initial conditions. We first need to find expressions for $y'(t)$ and $y''(t)$

$$y'(t) = c_1\, e^t \; - \; c_2 \sin t \; + \; c_3 \cos t + (t+1)e^t \quad .$$

$$y''(t) = c_1\, e^t \; - \; c_2 \cos t \; - \; c_3 \sin t + (t+2)e^t \quad .$$

Now plug-in $t = 0$:

$$y(0) = c_1\, e^0 \; + \; c_2 \cos 0 \; + \; c_3 \sin 0 + 0 \cdot e^0 \quad .$$

$$y'(0) = c_1\, e^0 \; - \; c_2 \sin 0 \; + \; c_3 \cos 0 + (0+1) \cdot e^0 \quad .$$

$$y''(0) = c_1\, e^0 \; - \; c_2 \cos 0 \; - \; c_3 \sin 0 + (0+2) \cdot e^0 \quad .$$

Simplifying:

$$y(0) = c_1 + c_2 \quad .$$

$$y'(0) = c_1 \; + \; c_3 + 1 \quad .$$

$$y''(0) = c_1 \; - \; c_2 \; + \; 2 \quad .$$

Using the initial values $y(0) = 0 \quad , \quad y'(0) = 2 \quad , \quad y''(0) = 2$, we get a **system** of three **linear equations** with three **unknowns**.

$$c_1 + c_2 = 0 \quad , \quad c_1 + c_3 + 1 = 2 \quad , \quad c_1 - c_2 + 2 = 2 \quad .$$

Simplifying:

$$c_1 + c_2 = 0 \quad , \quad c_1 + c_3 = 1 \quad , \quad c_1 - c_2 = 0 \quad .$$

From the first and third equations we get $c_1 = 0, c_2 = 0$. From the second equation, we get $0 + c_3 = 1$ so $c_3 = 1$. So we found out that

$$c_1 = 0 \quad , \quad c_2 = 0 \quad , \quad c_3 = 1 \quad .$$

Going back the above general solution and replacing $c_1, c_2, c_3$ by their numerical values, we get

$$y(t) = 0 \cdot e^t \; + \; 0 \cdot \cos t \; + \; 1 \cdot \sin t + te^t = \sin t + te^t \quad .$$

**Comment**: This is a very long **multi-step** problem.

**2.** Verify that the given function $y_1(x)$ is a solution of the given diff.eq. , then find a second solution of the given differential equation, then write down the general solution.

$$(x-1)\,y''(x) - x\,y'(x) + y(x) = 0 \quad , \quad x > 1 \quad ; \quad y_1(x) = e^x \quad .$$

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**Ans.**: $y(x) \;= c_1\,e^x \;+\; c_2\,x \quad (c_1, c_2$ arbitrary numbers).

**Type:** Infinite family of functions of $x$.

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First we need to check that $y_1(x) = e^x$ is indeed a solution

$(x-1)\,(e^x)''(x) - x\,(e^x)' + e^x \;= 0 = (x-1)\,e^x - x\,e^x + e^x \;= e^x(x-1-x+1) = e^x \cdot 0 = 0 \quad .$

(Yea!).

We now need to use the method of **reduction of order**. First divide the diff. eq. by the coefficient of $y''(x)$, so that it becomes 1.

$$y''(x) - \frac{x}{x-1}\,y'(x) + \frac{1}{x-1}y(x) = 0 \quad .$$

This gives us

$$p(x) = -\frac{x}{x-1} \quad .$$

Recall that that $y_2(x) = y_1(x) \cdot v(x)$ where $v(x)$ is a solution of the diff.eq.

$$y_1(x)v''(x) + (2y_1'(x) + p(x)y_1(x))v'(x) = 0 \quad .$$

So, in this problem,

$$e^x v''(x) + (2(e^x)' - \frac{x}{x-1} \cdot e^x)v'(x) = 0 \quad .$$

Simplifying

$$e^x v''(x) + e^x(2 - \frac{x}{x-1})v'(x) = 0 \quad ,$$

Dividing by $e^x$:

$$v''(x) + (2 - \frac{x}{x-1})v'(x) = 0 \quad ,$$

and simplifying some more

$$v''(x) + (\frac{2(x-1) - x}{x-1})v'(x) = 0 \quad ,$$

so

$$v''(x) + (\frac{x-2}{x-1})v'(x) = 0 \quad .$$

Since $v(x)$ is missing, we call $v'(x) = u(x)$, getting a **first-order diff.eq.**

$$u'(x) + \frac{x-2}{x-1} \cdot u(x) = 0 \quad .$$

This diff. eq. can be either solved using the method of **Integrating factor**, or the method of **separation of variables**. Let's use the latter

$$\frac{du}{dx} = -\frac{(x-2)u}{x-1} \quad .$$

Separating the $u$-stuff from the $x$-stuff:

$$\frac{du}{u} = -\frac{x-2}{x-1}dx \quad .$$

Integrate both sides

$$\int \frac{du}{u} = -\int \frac{x-2}{x-1}dx \quad .$$

Recall from calc2 (writing $\frac{x-2}{x-1} = \frac{(x-1)-1}{x-1} = 1 - \frac{1}{x-1}$ )

$$\int \frac{x-2}{x-1}dx = \int \left(1 - \frac{1}{x-1}\right) dx = x - ln(x-1) \quad ,$$

(note that we **don't** need $+C$ at the present context, since we are happy with one answer, in other words, we make $C = 0$)

Of course

$$\int \frac{du}{u} = \ln u \quad .$$

So

$$\ln u = -x + \ln(x-1) \quad .$$

Exponentiating both sides:

$$e^{\ln u} = e^{-x+\ln(x-1)} = e^{-x}e^{\ln(x-1)} = e^{-x}(x-1) \quad .$$

We got

$$u(x) = (x-1)e^{-x} \quad .$$

But $v(x) = \int u(x)\,dx$, so

$$v(x) = \int (x-1)e^{-x} = (x-1)e^{-x} - \int (-e^{-x}) \cdot 1 = (x-1)e^{-x} + e^{-x} = xe^{-x} \quad .$$

(Integration by parts: $\int uv'\,dx = uv - \int u'v\,dx$). So, we get that the **second** fundamental solution is

$$y_2(x) = y_1(x) \cdot v(x) = e^x \cdot (xe^{-x}) = x \quad .$$

**Finally**, the **General solution** is

$$y(x) = c_1e^x + c_2x \quad .$$

**Comment**: This is a very long **multi-step** problem.

**3**. (10 pts. altogether, 5 each)
Classify the critical point $(0,0)$ as to type, and determine whether it is is stable, asymptotically stable, or unstable, for the following systems. Explain
**a.** (5 pts.)

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \mathbf{x}(t)$$

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**Ans.**: type: Saddle point ; stability status: Unstable

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We need to find the **eigenvalues** of the matrix. Let's first form the $r$-matrix, obtained by subtracting $r$ **only** from the diagonal entries

$$A_r = \begin{pmatrix} 2-r & 3 \\ 5 & 6-r \end{pmatrix} \quad .$$

Next take the **determinant**

$$\det A_r = (2-r)\cdot(6-r) - 3\cdot 5 = (r-2)(r-6) - 15 = r^2 - 8r + 12 - 15 = r^2 - 8r - 3 = 0 \quad .$$

Using the famous **quadratic** formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad ,$$

with $a = 1, b = -8, c = -3$, we get

$$r_1, r_2 = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-3)}}{2\cdot 1} = \frac{-(-8) \pm \sqrt{64+12}}{2} = \frac{-(-8) \pm \sqrt{76}}{2} = \frac{8 \pm \sqrt{4\cdot 19}}{2} = \frac{8 \pm 2\sqrt{19}}{2} = 4 \pm \sqrt{19}$$

(Note: MANY people messed up at this step. They forgot that minus times minus is plus and got (**erroneously!**) $\frac{8\pm\sqrt{64-12}}{2} = \frac{8\pm\sqrt{52}}{2}$ and this careless mistake changed the verdict about the type of the critical point). PLEASE DON'T BE CARELESS. OF COURSE, YOU HAD TO RUSH, but in the FINAL I will give you much more time to check and double-check your work).
So we have two **real** and **distinct** eigenvalues, of **opposite sign**. Hence the critical point ($(0,0)$ in our case) is a **saddle point**. and since (at least) one of the eigenvalues is positive, it is **unstable**.

**b.** (5 pts.)

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \mathbf{x}(t)$$

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**Ans.**: type: Improper Node. ; stability status: Unstable

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We need to find the **eigenvalues** of the matrix. Let's first form the $r$-matrix, obtained by subtracting $r$ **only** from the diagonal entries

$$A_r = \begin{pmatrix} 2-r & 1 \\ 0 & 2-r \end{pmatrix} \quad .$$

Next take the **determinant**

$$\det A_r = (2-r)\cdot(2-r) - 1\cdot 0 = (r-2)^2 - 0 = (r-2)^2$$

**Note**: **quite a few** people did (**erroneously!**) $\det A_r = (2-r)\cdot(2-r) - 1\cdot 0 = (r-2)^2 - 1 = (r-2)^2 - 1$. In other words, they "think" that $1 \cdot 0$ is 1. At your stage, such an error, even if you are in a rush, is not excusable.

Solving $(r-2)^2 = 0$ we get $r = 2, 2$, so we have **positive and repeated eigenvalues**. Since they are positive it is **unstable**, and it is definitely a **Node**. It remains to be seen whether or not it is proper or improper.
**Shortcut tip** (only applicable for $2 \times 2$ systems): It is always improper **unless** the matrix is a diagonal matrix, in which case it is proper).
The 'official' way is to find the **space of eigenvectors**, aka as **eigenspace**. The $r$- matrix, when $r = 2$ is

$$A_2 = \begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad .$$

So any eigenvector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad ,$$

that spells out to:

$$v_2 = 0 \quad , \quad 0 = 0 \quad .$$

Hence $v_2 = 0$ but $v_1$ is anything it wants. So every eigenvector is of the form

$$\begin{pmatrix} v_1 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ,$$

so we have **one** degree is freedom, so the **eigenspace** is **one-dimensional**, and our node is an **improper node**.

**Comment**: Many people got confused between the two types of ‘type’. In this question I asked for the type of the critical point (the origin) not, as usual, about the Type of the answer. Many people did both, and that’s OK.

**4**. (10 pts.) Use any (correct) method to solve the initial value system

$$x_1'(t) = x_1(t) + x_2(t) \quad , \quad x_2'(t) = -x_1(t) + 3x_2(t) \quad ; \quad x_1(0) = 1 \quad , \quad x_2(0) = 1 \quad .$$

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**Ans.**: $x_1(t) = e^{2t} \quad ; \quad x_2(t) = e^{2t} \quad .$

**Type(s):** A pair of (specific) functions of $t$.

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It turns out to be (a little) easier to **not** use matrices, but rather convert the system to a 2nd-order diff.eq. for one function.

From the first diff.eq., we can express $x_2(t)$ in terms of $x_1(t)$ (and its derivative)

$$x_2(t) = x_1'(t) - x_1(t) \quad .$$

Substituting this into the second diff. eq. we have

$$(x_1'(t) - x_1(t))' = -x_1(t) + 3(x_1'(t) - x_1(t)) \quad .$$

Simplifying:

$$x_1''(t) - x_1'(t) = -x_1(t) + 3x_1'(t) - 3x_1(t) \quad ,$$

simplifying more, bringing everything to the left

$$x_1''(t) - x_1'(t) + x_1(t) - 3x_1'(t) + 3x_1(t) = 0 \quad ,$$

collecting terms

$$x_1''(t) - 4x_1'(t) + 4x_1(t) = 0 \quad .$$

We get a **good-old** 2nd order diff. eq. with **constant coefficients** that is homog. We know that $x_1(0) = 1$, but in order to solve the IVP, we need to know $x_1'(0)$. From the first diff. eq. (plugging-in $t = 0$) we have

$$x_1'(0) = x_1(0) + x_2(0) = 1 + 1 = 2 \quad ,$$

hence we have to solve the IVP

$$x_1''(t) - 4x_1'(t) + 4x_1(t) = 0 \quad , \quad x_1(0) = 1 \quad , \quad x_1'(0) = 2 \quad .$$

The characteristic equation is $r^2 - 4r + 4 = 0$ so $(r-2)^2 = 0$ and we have **repeated roots** $r_1 = 2, r_2 = 2$, so the **general solution** is

$$x_1(t) = c_1 e^{2t} + c_2 t e^{2t} \quad .$$

Hence

$$x_1'(t) = 2c_1e^{2t} + c_2(1 + 2t)e^{2t} \quad .$$

Plugging-in $t = 0$:

$$x_1(0) = c_1 \quad .$$

Hence

$$x_1'(0) = 2c_1 + c_2 \quad .$$

We get the system of two equations with two unknowns

$$c_1 = 1 \quad , \quad 2c_1 + c_2 = 2$$

giving $c_1 = 1$ and $c_2 = 0$. Going back to the general solution, we get

$$x_1(t) = 1 \cdot e^{2t} + 0 \cdot e^{2t} = e^{2t} \quad .$$

So much for $x_1(t)$. But recall from way back above that

$$x_2(t) = x_1'(t) - x_1(t) \quad ,$$

so

$$x_2(t) = (e^{2t})' - e^{2t} = 2e^{2t} - e^{2t} = e^{2t} \quad .$$

**Comment**: This is a very long **multi-step** problem.

**5**. (10 pts.) Set-up but do not compute a template for a particular solution of the diff.eq.

$$y'''(t) - 3y''(t) + 3y'(t) - y(t) = e^t + e^{2t} + \cos t \quad .$$

(Hint: $(r-1)^3 = r^3 - 3r^2 + 3r - 1 \quad .$)

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**Ans.**: A template is (using $A_1$,$A_2$, etc.) $y(t) = A_1 t^3 e^t + A_2 e^{2t} + A_3 \cos t + A_4 \sin t$

**Type:** : template for a particular solution (also ok: "family of functions")

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Using the hint, we have a **triple** root $r = 1, r = 1, r = 1$, so the general solution of the homog. version is

$$c_1 e^t + c_2 t e^t + c_3 t^2 e^t \quad ,$$

whose **components** (aka **fundamental solutions**) are $\{e^t, te^t, t^2 e^t\}$.
Using the **principal of superposition** we can tackle each piece on the right side one-at-a-time.

The **default template** for the piece $e^t$ is

$$A_1 e^t \quad ,$$

alas, it conflicts with the set of fundamental solutions, since $e^t$ is a member (so if you try it out in the diff. eq. you would get nonsense).

Our next try is to **literally** multiply it by $t$ (**Warning**: Do not make it $(A_1 t + A_0)e^t$.

$$A_1 t e^t \quad ,$$

alas, the new template still overlaps with $\{e^t, te^t, t^2 e^t\}$. So **try again**

$$A_1 t^2 e^t \quad ,$$

alas, the new template still overlaps with $\{e^t, te^t, t^2 e^t\}$. So **try again**

$$A_1 t^3 e^t \quad ,$$

and finally, there is no overlap.

The default template for the second piece is

$$A_2 e^{2t} \quad ,$$

and since there is no overlap, we keep it. The default template for the third piece is

$$A_3 \cos t + A_4 \sin t \quad ,$$

and since there is no overlap, we keep it. **Superposing** we have that the following a good template.

$$y_P(t) = A_1 t^3 e^t + A_2 e^{2t} + A_3 \cos t + A_4 \sin t \quad .$$

**Comment**: This kind of question is a **piece of cake** if you know what you are doing.

**6**. (10 pts.) **Using Variation of Parameters** (no credit for other methods!), first find a particular solution of

$$y''(t) - 5y'(t) + 4y(t) = 3e^{4t} \quad ,$$

then use it to find the general solution, and finally, use the latter to solve the initial value problem

$$y''(t) - 5y'(t) + 4y(t) = 3e^{4t} \quad ; \quad y(0) = 0 \quad , \quad y'(0) = 1.$$

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**Ans.**: $y(t) = te^{4t}$ .

**Type:** (specific) function of $t$ .

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We first must find a basis of **fundamental solutions** for the homog. version

$$y''(t) - 5y'(t) + 4y(t) = 0 \quad ,$$

The **characteristic equation** is

$$r^2 - 5r + 4 = 0 \quad .$$

Factoring: $(r-1)(r-4) = 0$ giving the roots $r = 1$ and $r = 4$. implying that

$$y_1(t) = e^t \quad , \quad y_2(t) = e^{4t} \quad .$$

The method of **variation of parameters** tells you that a particular solution is

$$y_P(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \quad ,$$

where $v_1(t), v_2(t)$ are functions of $t$ **to be determined**. They satisfy the system

$$v_1'(t)y_1(t) + v_2'(t)y_2(t) \;=\; 0$$

$$v_1'(t)y_1'(t) + v_2'(t)y_2'(t) \;=\; g(t) \quad ,$$

where $g(t) = 3e^{4t}$.

Spelling it out

$$v_1'(t)e^t + v_2'(t)e^{4t} \;=\; 0$$

$$v_1'(t)e^t + v_2'(t)4e^{4t} \;=\; 3e^{4t} \quad ,$$

From the first equation we get
$$v_1'(t) = -e^{3t} v_2'(t) \quad .$$
Plugging-in into the second equation
$$(-e^{3t}) v_2'(t) e^t + v_2'(t) 4e^{4t} \;=\; 3e^{4t} \quad ,$$
simplifying
$$(-e^{4t}) v_2'(t) + v_2'(t) 4e^{4t} \;=\; 3e^{4t} \quad ,$$
Dividing by $e^{4t}$, and simplifying:
$$3v_2'(t) \;=\; 3 \quad ,$$
so
$$v_2'(t) = 1 \quad .$$
Going back to $v_1'(t)$, we have
$$v_1'(t) = -e^{3t} v_2'(t) = -e^{3t} \quad .$$
Integrating, we have
$$v_2(t) = t \quad , \quad v_1(t) = -\frac{1}{3} e^{3t} \quad .$$
(Note that we don't care about the $+C$ in this step). So a **particular solution** is
$$y_P(t) = v_1(t) y_1(t) + v_2(t) y_2(t) = te^{4t} - \frac{1}{3} e^{3t} e^t = te^{4t} - \frac{1}{3} e^{4t} \quad ,$$
but the second piece **overlaps** with the general solution of the homog. version, and hence can be discarded, and a **better** particular solution is:
$$y_P(t) = te^{4t} \quad .$$
So the **general solution** of our diff. eq. is
$$y(t) = c_1 e^t + c_2 e^{4t} + te^{4t} \quad .$$
In order to solve the IVP, we need
$$y'(t) = c_1 e^t + 4c_2 e^{4t} + (1 + 4t) e^{4t} \quad .$$
Plugging-in $t = 0$:
$$y_(0) = c_1 e^0 + c_2 e^0 + 0 \cdot e^{40} = c_1 + c_2 \quad .$$
$$y'(0) = c_1 e^0 + 4c_2 e^0 + (1 + 4 \cdot 0) e^0 = c_1 + 4c_2 + 1 \quad .$$
Using the initial conditions $y(0) = 0, y'(0) = 1$. We have to solve the system
$$c_1 + c_2 = 0 \quad , \quad c_1 + 4c_2 + 1 = 1 \quad ,$$
in other words
$$c_1 + c_2 = 0 \quad , \quad c_1 + 4c_2 = 0 \quad ,$$
whose solution is $c_1 = 0, c_2 = 0$. Going back to the general solution we get
$$y(t) = 0 \cdot e^t + 0 \cdot e^{4t} + te^{4t} = te4t \quad .$$

**Comment**: This problem is easier to solve using the 'usual' way for solving inhomog. second-order linear differential equations with constant coefficients. If you are not told specifically to use "Variation of parameters" then, if applicable you are better off using the former method.

**7**. (10 pts.) Solve the initial value problem

$$y'''(t) - 10y''(t) + 25y'(t) = 25 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad , \quad y''(0) = 0 \quad .$$

---

**Ans.**: $y(t) = t$ .

**Type:** Function of $t$.

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We first must find the **general solution** of the **homog. version** (ignoring the initial conditions until later)

$$y'''(t) - 10y''(t) + 25y'(t) = 0 \quad .$$

The **characteristic equation** is

$$r^3 - 10r^2 + 25r = 0 \quad .$$

Factoring

$$r(r-5)^2 = 0 \quad ,$$

so the roots are $0, 5, 5$ (i.e. 5 is repeated). The general solution of the homog. version is

$$y(t) = c_1 e^{0t} + c_2 e^{5t} + c_3 t e^{5t} = c_1 + c_2 e^{5t} + c_3 t e^{5t} \quad .$$

Next it is time to look for a **particular solution**. The **default template** is

$$y_P(t) = A_1 \quad ,$$

alas it overlaps with the general solution (and trying it out would lead to nonsense: $0 = 25$). So the **updated template** is

$$y_P(t) = A_1 t \quad .$$

We have to substitute this **trial solution** into the diff. eq. First, we have to find the derivatives

$$y_P(t) = A_1 t \quad , \quad y_P'(t) = A_1 \quad , \quad y_P''(t) = 0 \quad , \quad y_P'''(t) = 0 \quad ,$$

So

$$A_1(0 - 10 \cdot 0 + 25 \cdot 1) = 25$$

$$25A_1 = 25 \quad ,$$

so $A_1 = 1$, and a particular solution is

$$y_P(t) = 1 \cdot t = t \quad .$$

Hence the **general solution** is

$$y(t) = c_1 + c_2 e^{5t} + c_3 t e^{5t} + t \quad .$$

We now take derivatives

$$y'(t) = 5c_2 e^{5t} + c_3(1+5t)e^{5t} + 1 \quad .$$

$$y''(t) = 25c_2 e^{5t} + c_3(10+25t)e^{5t} \quad .$$

Plugging-in $t = 0$:

$$y(0) = c_1 + c_2 e^0 + c_3 0 e^0 + 0 = c_1 + c_2 \quad .$$

$$y'(0) = 5c_2 e^0 + c_3(1 + 5 \cdot 0)e^0 + 1 = 5c_2 + c_3 \quad .$$

$$y''(0) = 25c_2 e^0 + c_3(10 + 25 \cdot 0)e^0 = 25c_2 + 10c_3 \quad .$$

So we to solve the system

$$c_1 + c_2 = 0 \quad , \quad 5c_2 + c_3 = 0 \quad , \quad 25c_2 + 10c_3 = 0 \quad ,$$

whose solution turns out to be (you do it!) $c_1 = 0, c_2 = 0, c_3 = 0$. Going back to the general solution, we have

$$y(t) = 0 + 0 \cdot e^{5t} + 0 \cdot te^{5t} + t \; = \; t \quad .$$

**8**. (10 pts.) Decide whether the following functions are linearly independent or linearly dependent. In the latter case find a linear relation among them.

$$y_1(t) = 2 + \cos t \quad , \quad y_2(t) = 3 + 2\cos t \quad , \quad y_3(t) = -1 + 3\cos t \quad .$$

---

**Ans.**: They are **linearly dependent**. A linear relation is

$$11y_1(t) - 7y_2(t) + y_3(t) = 0 \quad .$$

---

We are looking for a tentative linear relation

$$k_1 y_1(t) + k_2 y_2(t) + k_3 y_3(t) = 0 \quad ,$$

If all we can come up with is the trivial solution $k_1 = 0, k_2 = 0, k_3 = 0$ then the verdict would be 'linearly independent', but if we can come up with not-all-zero solution it would be 'linearly dependent'.

So we are looking for **numbers** $k_1, k_2, k_3$, such that

$$k_1(2 + \cos t) + k_2(3 + 2\cos t) + k_3(-1 + 3\cos t) = 0 \quad ,$$

Distributing (aka as *opening-up parantheses*), we get

$$2k_1 + k_1 cost + 3k_2 + 2k_2 \cos t - k_3 + 3k_3 \cos t = 0 \quad ,$$

$$(2k_1 + 3k_2 - k_3) + (k_1 + 2k_2 + 3k_3)\cos t = 0 \quad .$$

Comparing coefficient of $\cos t$, and the free term, we have

$$2k_1 + 3k_2 - k_3 = 0 \quad , \quad k_1 + 2k_2 + 3k_3 = 0 \quad .$$

This is a (homog.) system with **three** unknowns, but only **two** equations, so we know *a priori* that there is a solution. Let's find it. Taking, for convenience $k_3 = 1$ (we can take anything except 0).

$$2k_1 + 3k_2 - 1 = 0 \quad , \quad k_1 + 2k_2 + 3 = 0 \quad .$$

So

$$2k_1 + 3k_2 = 1 \quad , \quad k_1 + 2k_2 = -3 \quad .$$

From the second equation, we get

$$k_1 = -3 - 2k_2 \quad .$$

Substituting into the first equation

$$2(-3 - 2k_2) + 3k_2 = 1 \quad ,$$

so

$$-6 - 4k_2 + 3k_2 = 1 \quad ,$$

$$-k_2 = 7 \quad ,$$

so $k_2 = -7$. Finally, by **back substitution**, $k_1 = -3 - 2 \cdot (-7) = -3 + 14 = 11$. We got

$$k_1 = 11 \quad , \quad k_2 = -7 \quad , \quad k_3 = 1 \quad .$$

Going back to the general template of a linear relation, we got that a linear relation is

$$11 \cdot y_1(t) + (-7) \cdot y_2(t) + 1 \cdot y_3(t) = 0 \quad .$$

Since we found a linear relation, it follows that the three functions are **linearly dependent**.

**Comment:** Quite a few people left the answer as

$$11y_1(t) - 7y_2(t) + y_3(t) \quad ,$$

i.e. they forgot to put $= 0$ at the end. I only took a little bit of points off, but conceptually it is a big mistake. In the Final exam I would take many more points off. a disembodied $11y_1(t) - 7y_2(t) + y_3(t)$ is not a 'relation', it is just a 'linear combination' of the functions $y_1(t), y_2(t), y_3(t)$.

**Another Comment**: There are many correct answers. Once you get one linear relation, you can many other ones by multiplying by any **non-zero** number. So

$$-11y_1(t) + 7y_2(t) - y_3(t) = 0 \quad ,$$

(where we multiplied by $-1$) and

$$33y_1(t) - 21y_2(t) + 3y_3(t) = 0 \quad ,$$

(where we multiplied by 3) are equally valid answers. But it always desirable to have as nice-looking as possible answer, and that is obtained by clearing all fractions, but otherwise having the smallest possible numbers (as the $k_1, k_2, k_3$), and making $k_1$ (or the first one that is not zero) positive.

**Yet another comment**: This kind of question is a bit abstract, but once you get used to it, is fairly fast. It would be a pity to miss it.

**9**. (10 pts.) Find the maximal open intervals for which there exists a unique solution for initial value problems for any number in that interval

$$(t^3 - 5t^2 + 4t)\, y'''(t) + (\cos t)\, y''(t) + (t^2 + 1)\, y(t) = e^t \quad .$$

---

**Ans.**: Open Intervals: $(-\infty, 0)$ , $(0, 1)$ , $(1, 4)$ , $(4, \infty)$ .

OR: $-\infty < t < 0$ , $0 < t < 1$ , $1 < t < 4$ , $4 < t < \infty$ .

---

Dividing by the coefficient of $y'''(t)$, the diff.eq. become

$$y'''(t) + \frac{\cos t}{t^3 - 5t^2 + 4t}\, y''(t) + \frac{t^2 + 1}{t^3 - 5t^2 + 4t}\, y(t) = \frac{e^t}{t^3 - 5t^2 + 4t} \quad .$$

The numerators are all well-behaved, continuous functions that never blow-up. The denominator in all the terms (except the first one) is

$$t^3 - 5t^2 + 4t = t(t^2 - 5t + 4) = t(t-1)(t-4) \quad .$$

This is zero when $t = 0$, $t = 1$, and $t = 4$. So the coefficients blow-up with $t = 0$, $t = 1$, $t = 4$. These are the **trouble-makers**. So as long as we avoid these, then we are safe. The open intervals that avoid these are $(-\infty, 0)$, $(0, 1)$, $(1, 4)$, $(4, \infty)$.

**Comment**: This kind of questions are really easy. What a shame to miss them!

**10**. (10 pts.) Solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}(t) \quad , \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad .$$

---

**Ans.**:

$$\mathbf{x}(t) \;== \begin{pmatrix} e^t \\ -\frac{3}{2}e^t + \frac{3}{2}e^t \cos 2t + e^t \sin 2t \\ e^t - e^t \cos 2t + \frac{3}{2}e^t \sin 2t \end{pmatrix}$$

**Type:** vector of (specific) functions

---

The $r$-matrix is

$$A_r \;=\; \begin{pmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{pmatrix}$$

The **determinant** is

$$\det A_r = (1-r) \cdot \det \begin{pmatrix} 2 & 1-r \\ 2 & 1-r \end{pmatrix} \;-\; 0 \cdot \det \begin{pmatrix} 2 & -2 \\ 3 & 1-r \end{pmatrix} \;+\; 0 \cdot \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= (1-r)((1-r)^2 - (-2)(2)) = (1-r)((r-1)^2 + 4) = (1-r)(r^2 - 2r + 5) \quad .$$

So one eigenvalue is $r = 1$, and the other two are $\frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = 1 \pm 2i$. This is a complex conjugate pair.
Next we have to find an eigenvector corresponding to $r = 1$. The $r$ matrix is now

$$A_1 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}$$

If $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a tentative eigenvecor corresponding to $r = 1$, we need

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad .$$

Spelling this out in everyday notation:

$$0 = 0 \quad , \quad 2v_1 - 2v_3 = 0 \quad , \quad 3v_1 + 2v_2 = 0 \quad .$$

So $v_3 = v_1$ and $v_2 = -\frac{3}{2}v_1$, getting that an eigenvector is

$$\begin{pmatrix} v_1 \\ -\frac{3}{2}v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix} \quad .$$

For convenience, we pick $v_1 = 2$ getting that an eigenvector corresponding to $r = 1$ is

$$\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \quad .$$

Next we have to deal with $r = 1 + 2i$.
The $r$-matrix, $A_r$ is now

$$\begin{pmatrix} 1-(1+2i) & 0 & 0 \\ 2 & 1-(1+2i) & -2 \\ 3 & 2 & 1-(1+2i) \end{pmatrix} = \begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix}$$

The corresponding eigenvector, let's call it again, $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, must satisfy

$$\begin{pmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad .$$

Spelling this out, we get the system

$$(-2i)v_1 = 0 \quad , \quad 2v_1 - 2iv_2 - 2v_3 = 0 \quad , \quad 3v_1 + 2v_2 - 2iv_3 = 0 \quad .$$

From the first equation we get $v_1 = 0$, so $-2iv_2 - 2v_3 = 0, 2v_2 - 2iv_3 = 0$. From either of them (they are equivalent) we get $v_2 = iv_3$. Taking, for convenience $v_3 = 1$ (you can take any **non-zero** number. We get that an eigenvector corresponding to $r = 1 + 2i$ is

$$\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \quad .$$

So a fundamental solution is, in complex language:

$$e^{(1+2i)t} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} \quad .$$

Since $e^{(1+2i)t} = e^{t+2it} = e^t e^{2it}$, and thanks to Euler, $e^{2it} = \cos 2t + i \sin 2t$, we get that a fundamental solution, still in complex language is

$$e^t \begin{pmatrix} 0 \\ i(\cos 2t + i \sin 2t) \\ \cos 2t + i \sin 2t \end{pmatrix} = e^t \begin{pmatrix} 0 \\ i \cos 2t - \sin 2t \\ \cos 2t + i \sin 2t \end{pmatrix} = e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + ie^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \quad .$$

So we got **two** fundamental solutions, by taking the **real part** and **imaginary part**:

$$e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} \quad , \quad e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \quad .$$

Combined with the above fundamental solution, corresponding to $r = 1$, we get that the **general solution** is

$$\mathbf{x}(\mathbf{t}) = c_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + c_3 e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} \quad .$$

Now it is time to take advantage of the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

$$\mathbf{x}(\mathbf{0}) = c_1 \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_1 \\ -3c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_1 \\ -3c_1 + c_3 \\ 2c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad .$$

So we have to solve the system of linear equations

$$2c_1 = 1 \quad , \quad -3c_1 + c_3 = 0 \quad , \quad 2c_1 + c_2 = 0 \quad .$$

From the first equation, we get $c_1 = \frac{1}{2}$, From the second $c_3 = \frac{3}{2}$ and from the third $c_2 = -1$. Going back the **general solution**, we get

$$\mathbf{x}(\mathbf{t}) = \frac{1}{2} e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} - e^t \begin{pmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{pmatrix} + \frac{3}{2} e^t \begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} = \begin{pmatrix} e^t \\ -\frac{3}{2} e^t \\ e^t \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \sin 2t \\ -e^t \cos 2t \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{3}{2} e^t \cos 2t \\ \frac{3}{2} \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^t \\ -\frac{3}{2} e^t + \frac{3}{2} e^t \cos 2t + e^t \sin 2t \\ e^t - e^t \cos 2t + \frac{3}{2} \sin 2t \end{pmatrix} \quad .$$

**Note:** This problem is too long for a test. I did not try it out first myself. Sorry.