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MATH 244 (1-3), Solutions to Exam I for Dr. Z.'s Math 244(1-3), Fall 2016, Oct. 17, 2016, 8:40-10:00am, SEC 117

No Calculators! No Cheatsheets!
Write the final answer to each problem in the space provided. Incorrect answers (even due to minor errors) can receive at most one half partial credit, so please check and double-check your answers. Also indicate the type of the answer in the space provided (if applicable). Indicating the wrong type will automatically give you zero points.
Show your work! An answer without showing your work will get you zero points.
Note on grading: I announced that if the type is wrong you would get zero points, even if the answer is correct. I decided that this was too harsh, so I took points off for giving the wrong type, but still gave partial credit.

1. (10 pts.) Solve the initial value differential equation

$$y''(t) - 2\,y'(t) + 2\,y(t) = 0 \quad , \quad y(0) = 0 \quad , \quad y'(0) = 1 \quad .$$

Type of Answer (circle): **(a) function in explicit format** (b) function in implicit format (c) family of functions (d) Decision (e) Number (f) None of the above

Ans.: $y(t) = e^t \sin t$

The **characteristic equation** is

$$r^2 - 2r + 2 = 0$$

Using the quadratic formula

$$r_1, r_2 = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4}i}{2} = \frac{2 \pm 2i}{2} = 1 \pm i \quad .$$

So we have **complex roots** and $\lambda = 1$ and $\mu = 1$.
Hence the **general solution** is

$$y(t) = e^t \cdot (c_1 \cos t + c_2 \sin t) \quad .$$

Since we have an **initial value problem**, we have to find c_1, c_2 that would make them happy.
First we need to find $y'(t)$

$$y'(t) = (e^t \cdot (c_1 \cos t + c_2 \sin t))' = (e^t)' \cdot (c_1 \cos t + c_2 \sin t) + (e^t) \cdot (c_1 \cos t + c_2 \sin t)' =$$

$$(e^t) \cdot (c_1 \cos t + c_2 \sin t) + (e^t) \cdot (-c_1 \sin t + c_2 \cos t) \quad .$$

Plugging-in $t = 0$ in $y(t) = 0$ gives

$$0 = y(0) = e^0 \cdot (c_1 \cos 0 + c_2 \sin 0) = c_1 \quad ,$$

so $c_1 = 0$. Plugging-in $t = 0$ into $y'(t)$ gives

$$1 = y'(0) = (e^0) \cdot (c_1 \cos 0 + c_2 \sin 0) + (e^t) \cdot (-c_1 \sin 0 + c_2 \cos 0) \quad ,$$

So

$$1 = c_1 + c_2 \quad .$$

But we already know that $c_1 = 0$, so $c_2 = 1$. Going back to the general solution we get that

$$y(t) = e^t(0 \cdot \cos t + 1 \cdot \sin t) = e^t \sin t \quad ,$$

2. (10 pts.) Find the Wronskian, $W(f(t), g(t))$ of the following pair of functions:

$$f(t) = e^{t^2} \quad , \quad g(t) = te^{t^2} \quad .$$

Type of Answer (circle): **(a) function in explicit format** (b) function in implicit format (c) family of functions (d) Decision (e) Number (f) None of the above

Ans.: $W(f(t), g(t)) = e^{2t^2}$

$$W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t) \quad .$$

We have, by the **chain rule**

$$f'(t) = (e^{t^2})' = e^{t^2} \cdot (2t) = 2te^{t^2}$$

We also have, by the **produt rule** followed by the **chain rule**

$$g'(t) = (te^{t^2})' = t'e^{t^2} + t(e^{t^2})' = e^{t^2} + t(2te^{t^2}) = e^{t^2}(1 + 2t^2) \quad .$$

Going back to the Wronskian

$$W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t) = e^{t^2} \cdot e^{t^2}(1+2t^2) - 2te^{t^2} \cdot (te^{t^2}) = e^{2t^2}(1+2t^2-2t^2) = e^{2t^2} \quad .$$

Comments: Some people messed up the differentions, and some (very few) to my horror "simplified"
"$e^{t^2} \cdot e^{t^2} = e^{t^4}$". Please **watch out** not to make such gross algebra mistakes.

3. (10 pts.) First solve the initial value, first-order, differential equation,

$$t\,y'(t) + (t+2)\,y(t) = \frac{3t}{e^t} \quad , \quad y(1) = \frac{1}{e} \quad .$$

Having done that, find the value of that solution at $t = 2$, i.e. find $y(2)$.

Type of Answer (circle): (a) function in explicit format (b) function in implicit format (c) family of functions (d) Decision **(e) Number** (f) None of the above

Ans.: $y(2) = \frac{2}{e^2}$

This is a **first-order linear equation** requiring the method of **integrating factor**.
First divide the diff. eq. by the coefficient of $y'(t)$, getting the modified diff. eq.

$$y'(t) + \frac{t+2}{t}\,y(t) = 3e^{-t} \quad .$$

So $p(t) = \frac{t+2}{t} = 1 + \frac{2}{t}$. Integrating, we get $\int p(t)\,dt = \int (1 + \frac{2}{t})\,dt = t + 2\ln t = t + \ln t^2$.
Warning: This is not yet I(t)!
Hence, the intergating factor, $I(t)$ is

$$I(t) \;=\; e^{t+\ln t^2} = e^t \cdot e^{\ln t^2} = e^t \cdot t^2 = t^2 e^t \quad .$$

The **modified** right side, $g(t)$ is $3e^{-t}$
(**Warning:** some people took $g(t)$ as $\frac{3t}{e^t}$ (i.e. the origianl right side), watch out!)
Hence, the general solution is

$$y(t) = \frac{\int I(t)g(t)\,dt}{I(t)} = \frac{\int t^2 e^t 3e^{-t}\,dt}{t^2 e^t} = \frac{\int 3t^2\,dt}{t^2 e^t} = \frac{t^3 + C}{t^2 e^t} = te^{-t} + \frac{C}{t^2 e^t} \quad .$$

We still need to find C. Plugging-in $t = 1$, gives

$$\frac{1}{e} = y(1) = 1 \cdot e^{-1} + \frac{C}{1^2 e^1} \quad ,$$

so $C = 0$. Going back to the general solution, we have that the solution of the initial value problem is

$$y(t) = te^{-t} + \frac{0}{t^2 e^t} = \frac{t}{e^t} \quad .$$

Finally, we plug-in $t = 2$ to get $y(2) = \frac{2}{e^2}$.

4. (10 pts.) Use the **Improved Euler method** to find an approximate value for $y(1.1)$, if $y(x)$ is the solution of the initial value problem differential equation

$$y' = x + y \quad , \quad y(1) = 0 \quad ,$$

using mesh-size $h = 0.1$.
Reminder: $x_n = x_0 + nh$, $y_n^* = y_{n-1} + hf(x_{n-1}, y_{n-1})$,
$y_n = y_{n-1} + \frac{h}{2} \cdot (f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*))$.

Type of Answer (circle): (a) function in explicit format (b) function in implicit format (c) family of functions (d) Decision **(e) Number** (f) None of the above

Ans.: According to the Improved Euler Method (with $h = 0.1$), $y(1.1)$ is approximately equal to: 0.11

Here $f(x, y) = x + y$, $h = 0.1$, and we only need **one step**.
$x_0 = 1$, $x_1 = 1.1$, $y_0 = 0$. $y_1^* = y_0 + hf(x_0, y_0) = 0 + (0.1) \cdot (1 + 0) = 0.1$.
Hence

$$y_1 = y_0 + \frac{0.1}{2} \cdot (f(x_0, y_0) + f(x_1, y_1^*)) = 0 + \frac{0.1}{2} \cdot ((0+1) + (1.1+0.1)) = \frac{0.1}{2} \cdot (2.2) = 0.11 \quad .$$

Comment: It is disappointing that quite a few people messed up this easy problem. It is very important to know how to follow, by hand, simple algorithms. Quite a few people messed-up the simple calculations. A fairly common error was

$$\frac{0.1}{2} \cdot ((0 + 1) + (1.1 + 0.1)) = \frac{0.1}{2} \cdot (0 + 1) + (1.1 + 0.1) \quad ,$$

using the "rule" $a(b + c) = ab + c$. **Please watch out!**.

5. (10 pts.) For the following first-order differential equation decide whether or not it is exact. If it is, solve it. Leave the answer in **implicit format**.

$$(2xy + y^3) + (x^2 + 3xy^2)\,\frac{dy}{dx} = 0$$

(Where, as usual, y is shorthand for a function of x. $y(x)$, and $\frac{dy}{dx}$ is shorthand for its first-derivative, aka as $y'(x)$, or y' for short.)

Type of Answer (circle): (a) function in explicit format (b) function in implicit format **(c) family of functions** (d) Decision (e) Number (f) None of the above

Ans.: $x^2y + xy^3 = C$, where C is an *arbitrary number* .

First we have to see whether this diff.eq. is **exact**. Here

$$M(x,y) = 2xy + y^3 \quad , \quad N(x,y) = x^2 + 3xy^2 \quad .$$

So $M_y = 2x + 3y^2$ and $N_x = 2x + 3y^2$. Since these are the same, it is indeed exact. We need of find a (calc3, multivariable) function $F(x,y)$, such that $F_x = M$ and $F_y = N$.

$$F = \int M(x,y)\,dx = \int (2xy + y^3)\,dx = x^2y + xy^3 + \phi(y) \quad ,$$

where $\phi(y)$ is **to be determined**. Equating F_y and N we get

$$F_y = x^2 + 3xy^2 + \phi'(y) = x^2 + 3xy^2 \quad ,$$

Hence $\phi'(y) = 0$ and we can take $\phi(y) = 0$. Hence, our *intermediate step* gives

$$F(x,y) = x^2y + xy^3 + 0 = x^2y + xy^3 \quad .$$

This is not yet the answer. The answer is obtained by setting $F(x,y) = C$, where C is an arbitrary constant. Getting a **family of functions**.
Comments: Quite a few people got the type as "function in implicit format". This is wrong. What we have here is a "a **family** of functions in implicit form". Since this was not one of the choices, the right choice is 'family of functions'.

6. (10 pts.) For the following diff. eq. determine the critical (equilibrium) solutions and decide, for each such solution, whether it is asymptotically stable, asymptotically unstable, or asymptotically semi-stable.

$$\frac{dy}{dt} = y(y-1)^2 \quad .$$

Explain!

Ans.: First Equilibrium solution is $y = 0$, and it is (circle)
(a) asympototically stable **(b) asympototically unstable** (c) asympototically semi-stable

Second equilibrium solution is $y = 1$ and it is (circle)
(a) asympototically stable (b) asympototically unstable **(c) asympototically semi-stable**

Setting the right side to 0 and solving for y gives two equilibrium solutions.
$y = 0$ and $y = 1$.

Regarding $y = 0$,
If you plug-in $y = -0.1$ the slope is **negative** hence it is **unstable from below** (it runs away from the line $y = 0$)
If you plug-in $y = 0.1$ the slope is **posive** hence it is **unstable from above** (it runs away from the line $y = 0$)
Hence it is **asymptotically unstable**.

Regarding $y = 1$,
If you plug-in $y = 0.9$ the slope is **positive** hence it is **stable from below** (it runs towards from the line $y = 1$)
If you plug-in $y = 1.1$ the slope is **posive** hence it is **unstable from above** (it runs away from the line $y = 0$)
Hence it is **asymptotically semi-stable**.

7. (10 pts.) Find the maximal open interval for which the following second-order diff.eq. initial value problem is guaranteed to have a unique solution. **Explain!**

$$(t-3)(t+4)\cdot y''(t)+\tan t\cdot y'(t)+e^t\cdot y(t)=t^2 \quad , \quad y(0)=1 \quad , \quad y'(0)=2.$$

[Warning: unlike $\sin t$ and $\cos t$, that never blow-up, $\tan t$ does sometimes.]

Type of Answer (circle): (a) function in explicit format (b) function in implicit format (c) family of functions (d) Decision (e) Number **(f) Open interval on the t-axis** (g) None of the above

Ans.: $-\frac{\pi}{2}<t<\frac{\pi}{2}$

Dividing by the coefficient of $y''(t)$, our diff. eq. becomes

$$y''(t)+\frac{\tan t}{(t-3)(t+4)}\cdot y'(t)+\frac{e^t}{(t-3)(t+4)}\cdot y(t)=\frac{t^2}{(t-3)(t+4)} \quad , \quad y(0)=1 \quad , \quad y'(0)=2.$$

Obviosuly $t=3$ and $t=-4$ are 'trouble-makers', but $\tan t$ blows up sooner! its 'trouble-spots' are at $t=-\frac{\pi}{2}$ and $t=\frac{\pi}{2}$, so the largest open inverval **containg** the initial time, $t=0$ is $-\frac{\pi}{2}<t<\frac{\pi}{2}$.
Comment: Quite a few people realized that $t=\frac{\pi}{2}$ is a blow-up point, but forgot that so is $t=-\frac{\pi}{2}$, and gave as the answer $-4<t<\frac{\pi}{2}$. being nice, I gave them 5 out 10 points.

8. (10 pts.) Find an equation of the curve that passes through the point $(0,0)$ and whose slope at (x,y) is $(\sin x)\cdot(\sec y)$. You may leave your answer either in implicit or explicit format (using, if necessary, inverse-trig functions).

Type of Answer (circle): (a) function in explicit format **(b) equation of a curve in THE 2-dimensions xy-plane** (c) family of functions (d) Decision (e) Number (f) None of the above

Ans.: $\sin y + \cos x = 1$ (if you prefer implicit forma)
Or
$y = arcsin(1-\cos x)$ (if you prefer explicit forma)

This calls for the method of **separation of variables** (already covered in calc2!). Since slope is derivative, we have the diff. eq.

$$\frac{dy}{dx} = (\sin x)\cdot(\sec y) \quad .$$

Hence

$$\frac{dy}{\sec y} = \sin x\, dx \quad .$$

But $\frac{1}{\sec y} = \cos y$, so we get

$$\cos y\, dy = \sin x\, dx \quad .$$

Integrating, we get

$$\int \cos y\ , dy \ = \ \int \sin x\, dx \quad .$$

$$\sin y = -\cos x + C \quad .$$

Since the curve must pass through the point $(x,y) = (0,0)$, we substitute $x = 0$, $y = 0$ into the above equation, getting

$$\sin 0 = -\cos 0 + C \quad .$$

$$0 = -1 + C \quad ,$$

so $C = 1$. Going back,

$$\sin y = -\cos x + 1 \quad ,$$

or

$$\sin y + \cos x = 1 \quad .$$

If you like explicit format, solve for y and get

$$y = \arcsin(1-\cos x) \quad .$$

9. (10 pts.) Find the general solution of the differential equation

$$y''(t) + y'(t) - 2\,y(t) = 0 \quad .$$

Type of Answer (circle): (a) function in explicit format (b) function in implicit format **(c) family of functions** (d) Decision (e) Number (f) None of the above

Ans.: $y(t) = c_1 e^t + c_2 e^{-2t}$, where c_1, c_2 are **any** numbers.

The **characteristic equation** is

$$r^2 + r - 2 = 0 \quad .$$

Factorizing

$$(r+2)(r-1) = 0 \quad ,$$

so the two roots are $r_1 = 1$ and $r_2 = -2$. Hence the **general solution is**

$$y(t) = c_1 e^{1t} + c_2 e^{-2t} == c_1 e^t + c_2 e^{-2t}$$

10. (10 pts.) Find the Wronskian (up to an unknown constant in front) of **any** two solutions, $y_1(t)$ and $y_2(t)$ of the following diff.eq.

$$t^3 y''(t) + t^3 \cos t \; y'(t) + e^t \; y(t) = 0 \quad .$$

Type of Answer (circle): (a) function in explicit format (up to an unknown multiplicative constant c**)** (b) function in implicit format (c) family of functions (d) Decision (e) Number (f) None of the above

Ans.: $W(y_1(t), y_2(t)) = ce^{-\sin t}$, for some **unknown constant** c .

You first make the coeffient of $y''(t)$, 1, by dividing by t^3.

$$y''(t) + \cos t \; y'(t) + \frac{e^t}{t^3} \; y(t) = 0 \quad .$$

Here we have $p(t) = \cos t$. So $\int p(t)\, dt = \int \cos t \, dt = \sin t$.
The formula to use is $W = ce^{-\int p(t)\, dt}$, so we get

$$W = ce^{-\sin t} \quad ,$$

for some unknown number c.