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MATH 244 (1-3), Dr. Z., FINAL EXAM, Friday, Dec. 20, 2013, 4:00-7:00pm,  
PH-115

No Calculators! No Cheatsheets! Write the final answer to each problem in the space provided.

Do not write below this line (office use only)

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1. 20 (out of 20)

2. 20 (out of 20)

3. 20 (out of 20)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

8. 10 (out of 10)

9. 10 (out of 10)

10. 10 (out of 10)

11. 10 (out of 10)

12. 10 (out of 10)

13. 10 (out of 10)

14. 10 (out of 10)

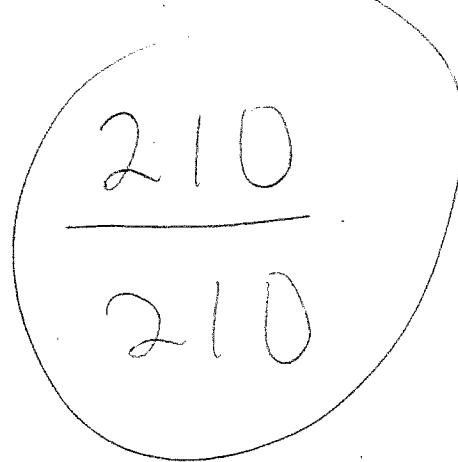
15. 10 (out of 10)

16. 10 (out of 10)

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total: \_\_\_\_\_ (out of 200)

You must check every answer (when applicable), and show your checking!



(20)

1. (20 pts., 10 each each)

(i) Seek a power series solution of the form  $y(x) = \sum_{n=0}^{\infty} a_n x^n$  to the differential equation

$$y''(x) + y'(x) - 2y(x) = 0$$

and set-up a recurrence for the coefficients  $a_n$ .

Ans.: Recurrence is:  $a_{n+2} = \frac{2}{(n+2)(n+1)} a_n - \frac{1}{(n+2)} a_{n+1}$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{m=0}^{m+2} (m+2)(m+1) a_{m+2} x^m + \sum_{m=1}^{\infty} (m+2) a_{m+2} x^{m+1} - 2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - 2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} [(m+2)(m+1) a_{m+2} + (m+1) a_{m+1} - 2a_m] x^m = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_{n+1} - 2a_n = 0$$

$$a_{n+2} = \frac{2a_n - (n+1)a_{n+1}}{(n+2)(n+1)} = \frac{2}{(n+2)(n+1)} a_n - \frac{1}{(n+2)} a_{n+1}$$

(ii) Using the recurrence that you found, find the first three terms of the fundamental solutions  $y_1(x)$  and  $y_2(x)$ .

Ans.:  $y_1(x) = 1 + x^2 - \frac{1}{3} x^3 + \dots ; \quad y_2(x) = x - \frac{1}{2} x^2 + \frac{1}{2} x^3 + \dots$

$n=0$

$$a_2 = \frac{2}{2} a_0 - \frac{1}{2} a_1 = a_0 - \frac{1}{2} a_1$$

$n=1$

$$a_3 = \frac{2}{6} a_1 - \frac{1}{3} a_2 = \frac{1}{3} a_1 - \frac{1}{3} (a_0 - \frac{1}{2} a_1) = \frac{1}{3} a_1 - \frac{1}{3} a_0 + \frac{1}{6} a_1 = \frac{1}{2} a_1 - \frac{1}{3} a_0$$

$\frac{2}{6}$   $\frac{3}{6}$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x + (a_0 - \frac{1}{2} a_1) x^2 + (-\frac{1}{3} a_0 + \frac{1}{2} a_1) x^3 + \dots$$

$$= a_0 + a_1 x + a_0 x^2 - \frac{1}{2} a_1 x^2 - \frac{1}{3} a_0 x^3 + \frac{1}{2} a_1 x^3 + \dots$$

$a_0 \text{ terms: } y_1(x) = 1 + x^2 - \frac{1}{2} x^3 + \dots$

2

2. (20 pts.) Use the Improved Euler method to find an approximate value for  $y(1.2)$  if  $y(x)$  is the solution of the initial value problem ode

$$y' = x + 2y \quad , \quad y(1) = 1 \quad ,$$

using mesh-size  $h = 0.1$ .

Reminder:

To solve the initial value problem

$$y' = f(\dot{x}, y) \quad , \quad y(x_0) = y_0$$

with mesh-size  $h$ , you define

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \dots$$

And compute, one-step-at-a-time

$$y_n^* = y_{n-1} + h f(x_{n-1}, y_{n-1}) \quad ,$$

$$y_n = y_{n-1} + h \frac{f(x_{n-1}, y_{n-1}) + f(x_n, y_n^*)}{2} \quad , \quad n = 1, 2, \dots$$

Then  $y_n$  is an approximation for  $y(x_n)$ . The smaller  $h$ , the better the approximation.

Ans.:  $y(1.2)$  is approximately equal to: 1.7547

$$1.1 + 2.676 = 3.776$$

$n$	$x_n$	$y_n$	$y_n^*$	$f(x_n, y_n)$	$f(x_n, y_n^*)$	$y_n^* = 1 + .1(3)$	$\frac{1.335}{1.712}$
0	1	1	N/A	$1+2(1)=3$	N/A	$= 1.3$	3.35
1	1.1	1.335	1.3	$1.1+2(1.335) = 3.770$	$1.1+2(1.3) = 3.7$	$y_1 = 1 + .1 \left( \frac{3 + 3.7}{2} \right)$	3.424
2	1.2	1.7547	1.7120		4.624	$y_1 = 1 + .1 \left( \frac{6.7}{2} \right) = 1 + .1(3.85) = 1 + .335 = 1.335$	4.624

$f(x_1, y_1) = 1.1 + 2(1.335) = 1.1 + 2.670$

$y_2^* = 1.335 + .1(1.1 + 2.670) = 3.770$

$= 1.335 + .1(3.770)$

$= 1.335 + .3770 = 1.7120$

$y_2 = 1.335 + .1 \left( \frac{3.770 + 4.624}{2} \right)$

$= 1.335 + .1 (4.197)$

$= 1.335 + .4197$

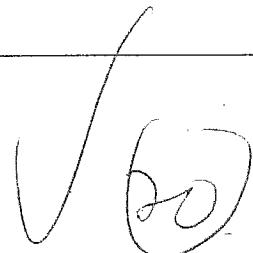
$= 1.7547$

$4.15 + .45 = 4.60$

3. (20 pts.) Solve the initial value problem

$$y'''(t) - y''(t) + y'(t) - y(t) = -1 \quad ; \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 2$$

Ans.:  $y(t) = e^t - \cos t + \sin t + 1$



$$y'''(t) - y''(t) + y'(t) - y(t) = -1$$

gen. sol.  
homog.

$$r^3 - r^2 + r - 1 = 0$$

$$(r-1)(r^2+1) = 0$$

$$r_1 = 1, r_2 = -i, r_3 = i$$

$$y_1(t) = e^t, y_2(t) = \cos t, y_3(t) = \sin t$$

particular  
solution

$$y_p(t) = A_0$$

$$0 - 0 + 0 - A_0 = -1$$

$$y'_p(t) = 0$$

$$A_0 = 1$$

$$y''_p(t) = 0$$

$$y_p(t) = 1$$

general solution  
to inhomogeneous  
diff. eq.:

$$y(t) = C_1 e^t + C_2 \cos t + C_3 \sin t + 1$$

$$y(0) = C_1 + C_2 + 1 = 1 \Rightarrow C_1 + C_2 = 0$$

$$C_1 = -C_2$$

$$y'(t) = C_1 e^t - C_2 \sin t + C_3 \cos t$$

$$C_1 = 1$$

$$y'(0) = C_1 + C_3 = 2 \Rightarrow 1 + C_3 = 2$$

$$C_3 = 1$$

$$y''(t) = C_1 e^t - C_2 \cos t - C_3 \sin t$$

$$y''(0) = C_1 - C_2 = 2$$

$$-C_2 - C_2 = 2$$

$$-2C_2 = 2$$

$$C_2 = -1$$

$$y(t) = e^t - \cos t + \sin t + 1$$

check:  $y'(t) = e^t + \sin t + \cos t$

$$y''(t) = e^t + \cos t - \sin t \rightarrow (e^t - \sin t - \cos t) - (e^t + \cos t - \sin t) + (e^t + \sin t + \cos t) - (e^t - \cos t + \sin t) = 0$$

$$y'''(t) = e^t - \sin t - \cos t \rightarrow (e^t - e^t + e^t - e^t) + (-\sin t + \sin t + \sin t - \sin t) + (-\cos t - \cos t + \cos t + \cos t) - 1 = -1$$

$$0 + 0 + 0 - 1 = -1 \checkmark$$

4. (20 pts.) Use any (correct) method to solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Ans.:  $\mathbf{x}(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



$$\det \begin{pmatrix} 0-r & 1 \\ 1 & 0-r \end{pmatrix} = (0-r)(0-r) - (1)(1) \\ = 0 - 0 - 0 + r^2 - 1 \\ r^2 - 1 = 0 \\ (r+1)(r-1) = 0 \\ r_1 = -1, r_2 = 1$$

$$r = -1 : \quad \begin{pmatrix} 0 - (-1) & 1 \\ 1 & 0 - (-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 + c_2 = 0 \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix} = c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$r = 1 : \quad \begin{pmatrix} 0 - (1) & 1 \\ 1 & 0 - (1) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-c_1 + c_2 = 0 \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ c_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$-c_1 + c_2 = 2 \quad \Rightarrow \quad c_2 + c_2 = 2$$

$$c_1 + c_2 = 0$$

$$c_1 = -c_2$$

$$c_1 + 1 = 0$$

$$c_1 = -1$$

$$x(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{check: } x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \checkmark$$

5. (20 pts.) Set-up but do not compute a template for a particular solution of the diff.eq.

$$y^{(5)}(t) - y^{(3)}(t) = t + e^t + 3e^{-t}$$

Ans.: A template is (using  $A_1, A_2$ , etc.)  $y(t) = (A_0 t^3 + A_1 t^4) + (B_0 t e^t) + (C_0 e^{-t})$

$$Y(t) = t^3(A_0 + A_1 t) + t(B_0 e^t) + t(C_0 e^{-t})$$

(20)

general solution  
to the homogeneous  $r^5 - r^3 = 0$   
diff eq:

$$r^3(r^2 - 1) = 0$$

$$r^3(r+1)(r-1) = 0$$

$$r_1 = -1, r_2 = 0, r_3 = 1, r_4 = 0, r_5 = 0$$

$$Y_1(t) = e^{-t}, Y_2(t) = e^{0t}, Y_3(t) = e^{1t} \quad \text{drats, overlap,}$$

$$Y_4(t) = t, Y_5(t) = t^2 \quad \text{Multiply by } t \text{ until no similar terms.}$$

6. (10 pts.) Find a particular solution of the diff.eq.

$$y'''(t) - y''(t) + y'(t) - y(t) = 20e^{3t}$$

Ans.:  $y(t) = e^{3t}$

$$\sqrt{6}$$

$$y'''(t) - y''(t) + y'(t) - y(t) = 20e^{3t}$$

General solution  
of the homogeneous:

$$r^3 - r^2 + r - 1 = 0$$

$$\begin{aligned} & r^2 + 1 \\ & r-1 \quad | \quad r^3 - r^2 + r - 1 \\ & \quad - r^3 - r^2 \\ & \quad \hline 0 + r - 1 \end{aligned} \Rightarrow (r-1)(r^2+1) = 0$$

$$r_2, r_3 = -\frac{1}{2} \pm \frac{\sqrt{1-4(-1)}}{2(-1)} = \pm i$$

$$r_1 = 1, r_2 = i, r_3 = -i$$

$$y_1(t) = e^t \quad y_2(t) = \cos t \quad y_3(t) = \sin t$$

Particular solution:

$$y(t) = A_0 e^{3t}$$

$$y'(t) = 3A_0 e^{3t}$$

$$y''(t) = 9A_0 e^{3t}$$

$$y'''(t) = 27A_0 e^{3t}$$

$$27A_0 e^{3t} - 9A_0 e^{3t} + 3A_0 e^{3t} - A_0 e^{3t} = 20 e^{3t}$$

$$27A_0 - 9A_0 + 3A_0 - A_0 = 20$$

$$20A_0 = 20$$

$$A_0 = 1$$

$$y(t) = e^{3t}$$

(no overlap w/ general solution to homogeneous, yeah!)

$$y'(t) = 3e^{3t}$$

$$y''(t) = 9e^{3t}$$

$$y'''(t) = 27e^{3t}$$

$$27e^{3t} - 9e^{3t} + 3e^{3t} - e^{3t} = 20e^{3t}$$

$$20e^{3t} = 20e^{3t} \checkmark$$

check:

7. (10 pts.) Solve the initial value problem

$$y''(t) + y'(t) - 2y(t) = 2t - 1, \quad y(0) = 1, \quad y'(0) = -2$$

Ans.:  $y(t) = \frac{2}{3}e^{-2t} + \frac{1}{3}e^t - t$

$\checkmark$  (10)

general solution  
of the homogeneous

$$y''(t) + y'(t) - 2y(t) = 2t - 1$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r_1 = -2, r_2 = 1$$

$$y_1(t) = e^{-2t}, y_2(t) = e^t$$

Particular solution

$$Y(t) = A_0 + A_1 t$$

$$Y'(t) = A_1$$

$$Y''(t) = 0$$

$$(0) + (A_1) - 2(A_0 + A_1 t) = 2t - 1$$

$$A_1 - 2A_0 - 2A_1 t = 2t - 1$$

$$(A_1 - 2A_0) = -1$$

$$-2A_0 = -1$$

$$A_0 = 0$$

$$Y(t) = -t$$

check:

$$Y'(t) = -1$$

$$Y''(t) = 0$$

$$0 - 1 - 2(-t) = 2t - 1$$

$$-1 + 2t = 2t - 1 \checkmark$$

General solution of  
the inhomogeneous

$$y(t) = C_1 e^{-2t} + C_2 e^t - t$$

$$y'(t) = -2C_1 e^{-2t} + C_2 e^t - 1$$

$$y(0) = C_1 + C_2 = 1 \implies C_2 = 1 - C_1$$

$$y'(0) = -2C_1 + C_2 - 1 = -2$$

$$-2C_1 + (1 - C_1) - 1 = -2$$

$$-2C_1 + 1 - C_1 - 1 = -2$$

$$-3C_1 = -2$$

$$C_1 = \frac{2}{3}$$

$$\frac{2}{3} + C_2 = 1$$

$$C_2 = \frac{1}{3}$$

$$y(t) = \frac{2}{3}e^{-2t} + \frac{1}{3}e^t - t$$

$$y'(t) = -\frac{4}{3}e^{-2t} + \frac{1}{3}e^t - 1$$

$$y(0) = \frac{2}{3} + \frac{1}{3} = 1 \checkmark$$

$$y'(0) = -\frac{4}{3} + \frac{1}{3} - 1 = -2 \checkmark$$

8. (10 pts.) Decide whether the following functions are linearly independent or linearly dependent. Explain!

$$y_1(t) = t, \quad y_2(t) = t^2, \quad y_3(t) = t^3$$

$$K_1(t) + K_2(t^2) + K_3(t^3) = 0 \quad (\text{Equation 1})$$

$$K_1t + K_2t^2 + K_3t^3 = 0$$

$$K_1 = 0 \quad K_2 = 0 \quad K_3 = 0$$

These functions are linearly independent because there aren't any non-zero values for  $K_1$ ,  $K_2$ , and  $K_3$  that make Equation 1 true.

✓ (10)

9. (10 pts.) Find the general solution of the differential equation

$$y''(t) + y(t) = 2e^t$$

Ans.:  $y(t) = C_1 \cos t + C_2 \sin t + e^t$

(where  $C_1, C_2$  are arbitrary constants)

10

$$y''(t) + y(t) = 2e^t$$

general solution  
of the homogeneous:  $r^2 + 1 = 0$   $\Rightarrow r_1, r_2 = -i \pm i$

$$r_1, r_2 = -b \pm \sqrt{\frac{b^2 - 4ac}{2a}} = -0 \pm \sqrt{\frac{0^2 - 4(1)(1)}{2(1)}} = \frac{\pm 2i}{2} = \pm i$$

$$\begin{aligned} Y_1(t) &= e^{0t} \cos t \\ Y_1(t) &= \cos t \end{aligned}$$

$$\begin{aligned} Y_2(t) &= e^{0t} \sin t \\ Y_2(t) &= \sin t \end{aligned}$$

Particular Solution:  $Y(t) = A_0 e^t$

$$Y'(t) = A_0 e^t$$

$$Y''(t) = A_0 e^t$$

$$e^{-t} [A_0 e^t + A_0 e^t] = 2e^t$$

$$A_0 + A_0 = 2$$

$$2A_0 = 2$$

$$A_0 = 1$$

$$Y(t) = (1)e^t = e^t$$

check:  $e^t + e^t = 2e^t \checkmark$

General Solution:  $y(t) = C_1 \cos t + C_2 \sin t + e^t$   
of the inhomogeneous

ANSWER:  $y'(t) = -C_1 \sin t + C_2 \cos t + e^t$

$$y''(t) = -C_1 \cos t - C_2 \sin t + e^t$$

$$(-C_1 \cos t - C_2 \sin t + e^t) + (C_1 \cos t + C_2 \sin t + e^t) = 2e^t$$

$$2e^t = 2e^t \checkmark$$

10. (10 pts.) Verify that the given solution  $y_1(t)$  is indeed a solution of the given diff.eq. Then Find a second solution of the given differential equation. Then write down the general solution.

$$t^2 y''(t) + 2t y'(t) - 2y(t) = 0, \quad t > 0; \quad y_1(t) = t$$

**Reminder:** Suppose that you have a *general* second-order homog. linear diff.eq. (not necessarily with constant coefficients), that can always be written (by dividing by the coefficient of  $y''(t)$ , if it is not already 1):

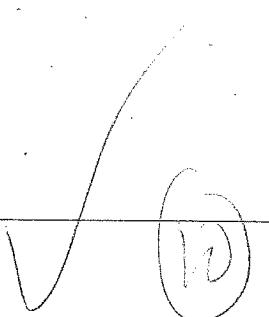
$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

Suppose that someone gave you one solution,  $y_1(t)$ , then you can find another (independent!) solution,  $y_2(t)$  (and hence all solutions, since the general solution would be  $c_1 y_1(t) + c_2 y_2(t)$ ), by writing

$$y(t) = v(t)y_1(t)$$

It is known that  $v(t)$  satisfies the diff.eq.

$$y_1(t)v''(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0$$



Ans.: General solution is  $y(t) = C_1 t + C_2 t^{-2}$

where  $C_1, C_2$  are arbitrary constants.

$$\begin{aligned} y(t) &= \frac{2t}{t^2} \\ &= \frac{2}{t} \end{aligned}$$

$$y_1(t) = t$$

$$t^2(0) + 2t(1) - 2(t) = 0$$

$$y_1'(t) = 1$$

$$0 + 2t - 2t = 0 \vee y_1(t) \text{ is a solution.}$$

$$y_1''(t) = 0$$

$$t v''(t) + \left[ 2(1) + \frac{2}{t} t \right] v'(t) = 0$$

$$\text{Check: } v_1'(t) = C_1 - 2C_2 t^{-3}$$

$$tv''(t) + [4]v'(t) = 0$$

$$v_1''(t) = 6C_2 t^{-4}$$

$$v''(t) + \frac{4}{t} v'(t) = 0$$

$$t^2(6C_2 t^{-4}) + 2t(C_1 - 2C_2 t^{-3}) - 2(C_1 t + C_2 t^{-2}) = 0$$

$$v'(t) + \frac{4}{t} v(t) = 0$$

$$\frac{6C_2}{t^2} + 2C_1 t - \frac{4C_2}{t^3} - 2C_1 t - \frac{2C_2}{t^2} = 0$$

$$I(t) = e^{\int \frac{4}{t} dt} = e^{4 \ln t} = e^{\ln t^4} = t^4$$

$$\left( \frac{6C_2}{t^2} - \frac{4C_2}{t^3} - \frac{2C_2}{t^2} \right) + (2C_1 t - 2C_1 t) = 0$$

$$t^4 v'(t) + 4t^3 v(t) = 0$$

$$0 + 0 = 0 \vee$$

$$\int [t^4 v(t)] dt = 0$$

$$t^4 v(t) = C \quad \text{let } C = 1$$

$$v(t) = \frac{1}{t^4} = t^{-4}$$

$$v(t) = \int v(t) dt = \frac{1}{3} t^{-3}$$

$$y_2(t) = y_1(t)v(t) = t \left( -\frac{1}{3} \frac{1}{t^3} \right) = -\frac{1}{3} \frac{1}{t^2}$$

11. (10 pts.) Find the general solution of the following diff. eq.

$$y''(t) - 10y'(t) + 25y(t) = 0$$

Ans.:  $y(t) = C_1 e^{5t} + C_2 t e^{5t}$

(where  $C_1, C_2$  are arbitrary constants)

✓ (10)

$$y''(t) - 10y'(t) + 25y(t) = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r-5)(r-5) = 0$$

$$r_1 = 5 = r_2$$

$$y(t) = C_1 e^{5t} + C_2 t e^{5t}$$

check:

$$\begin{aligned}y'(t) &= 5C_1 e^{5t} + 5C_2 t e^{5t} + C_2 e^{5t} = e^{5t}(5C_1 + 5C_2 t + C_2) \\y''(t) &= e^{5t}(5C_2) + 5(5C_1 + 5C_2 t + C_2)e^{5t} \\&= 5e^{5t}(C_2 + 5C_1 + 5C_2 t + C_2) \\&= 5e^{5t}(2C_2 + 5C_1 + 5C_2 t)\end{aligned}$$

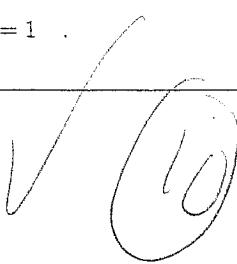
$$\begin{aligned}5e^{5t}(2C_2 + 5C_1 + 5C_2 t) - 10e^{5t}(5C_1 + 5C_2 t + C_2) + 25e^{5t}(C_1 + C_2 t) &= 0 \\(10C_2 + 25C_1 + 25C_2 t) - 50C_1 - 50C_2 t - 10C_2 + 25C_1 + 25C_2 t &= 0 \\(10C_2 - 10C_2) + (25C_1 + 25C_1 - 50C_1) + (25C_2 t - 50C_2 t + 25C_2 t) &= 0\end{aligned}$$

$$0 + 0 + 0 = 0 \quad \checkmark$$

12. (10 pts.) Solve the following initial value problems. Leave the answer in implicit form.

$$(3x^2 + y) + (3y^2 + x)y' = 0, \quad y(1) = 1$$

Ans.:  $3 = x^3 + y^3 + xy$



$$M = (3x^2 + y)$$

$$N = (3y^2 + x)$$

$$M_y = 1$$

$$N_x = 1$$

$$M_y = N_x \text{ so exact.. yeah!}$$

$$F = \int M dx = \int (3x^2 + y) dx$$

$$F = x^3 + xy + \phi(y)$$

$$F_y = N$$

$$x + \phi'(y) = 3y^2 + x$$

$$\int \phi'(y) dy = \int 3y^2 dy$$

$$\phi(y) = y^3 + C$$

$$F(x, y) = x^3 + xy + y^3 + C$$

$$C = x^3 + y^3 + xy$$

$$C = (1)^3 + (1)^3 + (1)(1)$$

$$C = 1 + 1 + 1 = 3$$

$$3 = x^3 + y^3 + xy$$

$$\begin{array}{r} 3 \\ 17 \\ \times 15 \\ \hline 195 \\ + 170 \\ \hline 255 \end{array}$$

13. (10 pts.) For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y^2(y-1)(y+1)$$

$y(t) = -1$  is an asymptotically stable equilibrium  
 Ans.: Solution,  $y(t) = 0$  is an asymptotically semi-stable equilibrium solution,  $y(t) = 1$  is an asymptotically unstable equilibrium solution.

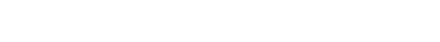
✓  
10

$$\frac{dy}{dt} = y^2(y-1)(y+1)$$

$$y^2 = 0 \quad y-1 = 0 \quad y+1 = 0$$

Equilibrium Solutions:  $y = -1, y = 0, y = 1$

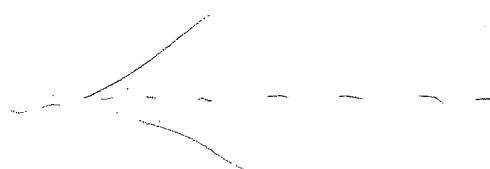
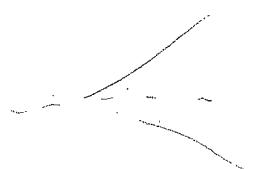
$f(t) = 1$  {

$f(-0.999) = (+)(-)(+)$ $= -$ so $\frac{dy}{dt} < 0$ so <small>stable from above</small>	
$f(-1.0001) = (+)(-)(-)$ $= +$ so $\frac{dy}{dt} > 0$ so <small>unstable from below</small>	

$f(t) = 0$  {

$f(-0.001) = (+)(-)(+)$ $= -$ so $\frac{dy}{dt} < 0$ <small>unstable from above</small>	
$f(0.001) = (+)(-)(+)$ $= +$ so $\frac{dy}{dt} > 0$ so <small>stable from below</small>	

$f(t) = -1$  {

$f(1.001) = (+)(+)(+)$ $= +$ so $\frac{dy}{dt} > 0$ so <small>unstable from above</small>	
$f(0.999) = (+)(-)(+)$ $= -$ so $\frac{dy}{dt} < 0$ so <small>stable from below</small>	

14. (10 pts.) Find an equation of the curve that passes through the point  $(1, 2)$  and whose slope at  $(x, y)$  is  $x^2/y^5$ .

Ans.: The curve is (in implicit form):

$$y^6 = 2x^3 + 62$$

✓ (10)

$$\frac{dy}{dx} = \frac{x^2}{y^5}$$

$$\begin{array}{r} 10 \\ 6 \\ -6 \\ \hline 04 \end{array}$$

$$\int y^5 dy = \int x^2 dx$$

$$\frac{1}{6}y^6 = \frac{1}{3}x^3 + C$$

$$2^3 = 8$$

$$\frac{1}{6}(2)^6 = \frac{1}{3}(1)^3 + C$$

$$2^4 = 16$$

$$\frac{64}{6} = \frac{1}{3} + C$$

$$2^5 = 32$$

$$\frac{64}{6} - \frac{2}{6} = C$$

$$C = \frac{62}{6} = \frac{31}{3}$$

$$\frac{1}{6}y^6 = \frac{1}{3}x^3 + \frac{31}{3}$$

$$y^6 = \frac{6}{3}x^3 + 6\left(\frac{31}{3}\right)$$

$$y^6 = 2x^3 + 62$$

15. (10 pts.) Using the method of integrating factor (no credit for other methods), solve the initial value problem

$$y'(t) - 2y(t) = e^{2t} \quad y(0) = 1$$

Ans.:  $y(t) = e^{2t}(t+1)$



$$I(t) = e^{\int p(t)dt} = e^{\int -2dt} = e^{-2t}$$

$$e^{-2t} [y'(t) - 2y(t)] = e^{2t}$$

$$e^{-2t} y'(t) - 2e^{-2t} y(t) = 1$$

$$\int [e^{-2t} y(t)] dt = 1 dt$$

$$e^{-2t} y(t) = t + C$$

$$y(t) = te^{2t} + Ce^{2t}$$

$$y(0) = 0 + C = 1$$

$$C=1$$

check:  $y(t) = te^{2t} + e^{2t} = e^{2t}(t+1)$

$$y(0) = e^{2 \cdot 0} = 1 \quad (0+1) = 1 \quad \checkmark$$

$$y(t) = \frac{\int I(t) q(t) dt}{I(t)} = \frac{\int e^{-2t} e^{2t} dt}{e^{-2t}}$$

$$= \frac{\int 1 dt}{e^{-2t}} = \frac{t + C}{e^{-2t}}$$

$$y(0) = \frac{0 + C}{e^{-2 \cdot 0}} = C = 1 \quad \checkmark$$

$$y(t) = e^{2t}(t+1)$$

table  
check:

NOT a typo

16. (10 pts.) Find out whether or not the function  $y(t) = t^4 + 1$  is a solution of the initial value differential equation

$$y^{(4)}(t) = 24, \quad y(1) = 2, \quad y'(1) = 4, \quad y''(1) = 12, \quad y'''(1) = 25$$

$$y(t) = t^4 + 1$$

$$y(1) = (1)^4 + 1 = 2 \checkmark$$

$$y'(t) = 4t^3$$

$$y'(1) = 4(1)^3 = 4 \checkmark$$

$$y''(t) = 12t^2$$

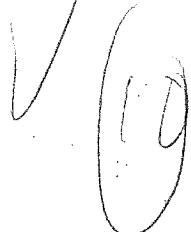
$$y''(1) = 12(1)^2 = 12 \checkmark$$

$$y'''(t) = 24t$$

$$y'''(1) = 24(1) = 24 \times$$

$$y^{(4)}(t) = 24$$

NO!



$y(t) = t^4 + 1$  is NOT a solution b/c this function gives  $y'''(1)=24$   
NOT  $y'''(1)=25$  as the initial conditions require.