

$$I = I -$$

$$I = (1 + \frac{dy}{dt}) - \frac{dy}{dt}$$

$$I = (1)I - (1)_{(0)}I$$

	$y(t) = e^t + 1$	$\frac{dy}{dt} = (e^t)_t + 1$
\wedge	$I = e^t y = (0) y$	$\frac{dy}{dt} = (1)_{(0)} y$
\wedge	$I = e^t y = (0)_{(1)} y$	$\frac{dy}{dt} = (1)_{(1)} y$
\wedge	$I = e^t y = (0)_{(2)} y$	$\frac{dy}{dt} = (1)_{(2)} y$
\wedge	$I = e^t y = (0)_{(3)} y$	$\frac{dy}{dt} = (1)_{(3)} y$
\wedge	$I = e^t y = (0)_{(4)} y$	$\frac{dy}{dt} = (1)_{(4)} y$
\wedge	$I = e^t y = (0)_{(5)} y$	$\frac{dy}{dt} = (1)_{(5)} y$
\wedge	$I = e^t y = (0)_{(6)} y$	$\frac{dy}{dt} = (1)_{(6)} y$
\wedge	$I = e^t y = (0)_{(7)} y$	$\frac{dy}{dt} = (1)_{(7)} y$
\wedge	$I = e^t y = (0)_{(8)} y$	$\frac{dy}{dt} = (1)_{(8)} y$
\wedge	$I = e^t y = (0)_{(9)} y$	$\frac{dy}{dt} = (1)_{(9)} y$
\wedge	$I = e^t y = (0)_{(10)} y$	$\frac{dy}{dt} = (1)_{(10)} y$

(1)

a solution
 $y(t) = e^t + 1$

$y(0) = 2$, $y'(0) = 1$, $y''(0) = 1$, $y'''(0) = 1$, $y^{(4)}(0) = 1$, $y^{(5)}(0) = 1$,
 $y^{(6)}(0) = 1$, $y^{(7)}(0) = 1$, $y^{(8)}(0) = 1$, $y^{(9)}(0) = 1$, $y^{(10)}(0) = 1$,

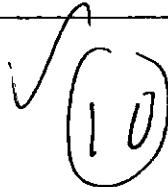
 $y^{(10)}(t) - y(t) = -1$

value differential equation
 16. (10 pts.) Find out whether or not the function $y(t) = e^t + 1$ is a solution of the initial

15. (10 pts.) Using the method of integrating factor (no credit for other methods), solve the initial value problem

$$y'(t) - y(t) = e^t, \quad y(0) = 0$$

Ans.: $y(t) = te^t$



$$I(t) = e^{\int p(t) dt}$$

$$y(t) = \frac{\int I(t) g(t) dt}{I(t)}$$

$$I(t) \cdot e^{\int -1 dt} = e^{-t}$$

$$y(t) = \frac{\int e^{-t} \cdot e^t dt}{e^{-t}}$$

$$y(t) = \frac{\int 1 dt}{e^{-t}} = \frac{t + C}{e^{-t}}$$

$$y(t) = te^t + Ce^t \rightarrow y(t) = te^t$$

$$y(0) = 0 + C = 0$$

$$C = 0$$

Check:

$$y(t) = te^t$$

$$y'(t) - y(t) = e^t$$

$$y'(t) = e^t + te^t$$

$$e^t + te^t - te^t = e^t$$

$$e^t = e^t \checkmark$$

$$y(0) = 0$$

$$0 \cdot e^0 = 0$$

$$0 = 0 \checkmark$$

14. (10 pts.) Find an equation of the curve that passes through the point $(1, 0)$ and whose slope at (x, y) is $-x/y$.

Ans.: The curve is (in implicit form): $\frac{x^2}{2} + \frac{y^2}{2} = \frac{1}{2}$

10

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y(1) = 0$$

$$\int y \cdot dy = \int -x \cdot dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{y^2}{2} + \frac{x^2}{2} = C$$

$$\frac{1}{2} + \frac{1}{2} = C \rightarrow C = \frac{1}{2}$$

Check:

$$\frac{x^2}{2} + \frac{y^2}{2} = \frac{1}{2}$$

$$\frac{y^2}{2} = \frac{1}{2} - \frac{x^2}{2}$$

$$\frac{1}{2} + \frac{0}{2} = \frac{1}{2}$$

$$\frac{y^2}{2} = \frac{1}{2} - \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2} \quad \checkmark$$

$$\frac{y^2}{2} = 0$$

$$y = 0$$

13. (10 pts.) For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y^5(y-1)^2$$

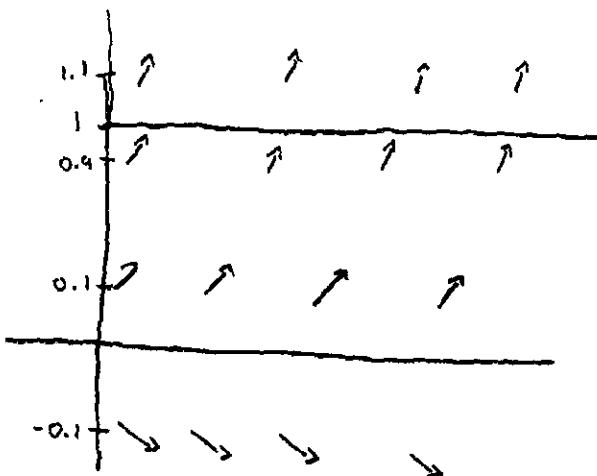
(10)

Ans.: $y=0 \rightarrow \text{unstable}$
 $y=1 \rightarrow \text{semi-stable}$

$$y^5(y-1)^2 = 0$$

$$y = 0, 0, 0, 0, 0$$

$$y = 1, 1$$



Check $\rightarrow y = -0.1$
 $(-0.1)^5 (-0.1-1)^2$
 $- + + = -$

$$y = 0.1$$

 $(0.1)^5 (0.1-1)^2$
 $+ + + = +$

$$y = 0.9$$

 $(0.9)^5 (0.9-1)^2$
 $+ * + = +$

$$y = 1.1$$

 $(1.1)^5 (1.1-1)^2$
 $+ * + = +$

12. (10 pts.) Solve the following initial value problems. Leave the answer in implicit form.

$$\frac{dy}{dx} = -\frac{2x+3y}{3x+2y}, \quad y(1) = 1$$

✓
10

Ans.: $x^2 + 3xy + y^2 = 5$

$$dy(3x+2y) + (2x+3y)dx = 0$$

$$(2x+3y)dx + (3x+2y)dy = 0$$

$$N_y = 3 \quad | \quad M_x = 3 \quad \text{exact !!}$$

$$\int (2x+3y)dx = x^2 + 3xy + \phi(y)$$

$$3x + \phi'(y) = 3x + 2y$$

$$\phi'(y) = 2y$$

$$\phi(y) = y^2$$

$$x^2 + 3xy + y^2 = C$$

$$1 + 3(1)(1) + 1 = C$$

$$1 + 3 + 1 = C$$

$$C = 5$$

Check:

$$x^2 + 3xy + y^2 = 5$$

$$1^2 + 3(1)(1) + 1^2 = 5$$

$$1 + 3 + 1 = 5$$

$$5 = 5 \checkmark$$

11. (10 pts.) Find the general solution of the following diff. eq.

$$y''(t) + 20y'(t) + 100y(t) = 0$$

Ans.: $y(t) = C_1 e^{-10t} + C_2 t e^{-10t}$

(where c_1, c_2 are arbitrary constants)

✓
10

$$r^2 + 20r + 100 = 0$$

$$(r+10)(r+10) = 0 \Rightarrow r^2 + 10r + 10r + 100 = 0 \\ r^2 + 20r + 100 = 0$$

$$r = -10, -10$$

$$r+10=0$$

$$y(t) = C_1 e^{-10t} + C_2 t e^{-10t}$$

$$r=-10$$

10. (10 pts.) Find the Wronskian (up to an unknown multiplicative constant, c) of *any* two solutions, $y_1(t), y_2(t)$ of the following diff.eq.

$$t y''(t) + t^3 y'(t) + (e^t + 1) y(t) = 0$$

Ans.: The Wronskian, $W(y_1(t), y_2(t))$ is (up to an unknown constant c):

$$c e^{-\frac{t^3}{3}}$$

Y
10

$$c e^{-\int p(t) dt}$$

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

$$c e^{-\int t^2 dt}$$

$$t y''(t) + t^3 y'(t) + (e^t + 1) y(t) = 0$$

$$c e^{-\frac{t^3}{3}}$$

$$y''(t) + t^2 y'(t) + \frac{(e^t + 1)}{t} y(t) = 0$$

$$p(t) = t^2$$

9. (10 pts.) Find the general solution of the differential equation

$$y''(t) + 4y(t) = 3 \sin t$$

Ans.: $y(t) = \sin t + C_1 \cos 2t + C_2 \sin 2t$
(where C_1, C_2 are arbitrary constants)

✓ (10)

Homogeneous:

$$r^2 + 4 = 0$$

$$r = \frac{0 \pm \sqrt{0^2 - 4(4)(0)}}{2}$$

$$= \frac{\pm \sqrt{-16}}{2} = \frac{\pm 4i}{2} = \pm 2i$$

$$Y_h = C_1 \cos 2t + C_2 \sin 2t$$

Particular:

$$y(t) = (A \sin t + B \cos t) =$$

$$y'(t) = A \cos t - B \sin t,$$

$$y''(t) = -A \sin t - B \cos t$$

$$-A \sin t - B \cos t + 4A \sin t + 4B \cos t = 3 \sin t$$

$$3A \sin t + 3B \cos t = 3 \sin t$$

$$3A = 3$$

$$A = 1$$

$$3B = 0$$

$$B = 0$$

$$y(t) = \sin t$$

Check Particular:

$$y(t) = \sin t$$

$$y'(t) = \cos t$$

$$y''(t) = -\sin t$$

$$-\sin t + 4\sin t = 3\sin t$$

$$3\sin t = 3\sin t \quad \checkmark$$

8. (10 pts) Compute the Wronskian of the three functions.

$$y_1(t) = t, \quad y_2(t) = t^2, \quad y_3(t) = t^3$$

Ans.: $W(y_1(t), y_2(t), y_3(t)) = 2t^5$

$\checkmark (10)$

$$W(y_1(t), y_2(t), y_3(t)) = \begin{vmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$t \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - 1 \begin{vmatrix} t^2 & t^3 \\ 2 & 6t \end{vmatrix} + 0 \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}$$

$$t(12t^2 - 6t^2) - 1(6t^3 - 2t^3) + 0$$

$$6t^3 - 4t^3 = 2t^3$$

$$\wedge \quad 0 = 0$$

$$0 = 1 - 1 - 1 + 1$$

$$0 = x - x - h + h$$

Check:

$$C = 0$$

$$C = 1 - 1 - 1 + 1$$

$$C = x - x - h + h$$

$$C + x + \frac{y}{x} = h + \sqrt{xy}$$

$$x \frac{dy}{dx} + y = h + h \frac{dy}{dx}$$

$$C = 0$$

$$C = 1 - 1 - 1 + 1$$

$$C = x - x - h + h$$

$$x - x = (x) \emptyset$$

$$1 - x = (x), \emptyset$$

$$(x)\emptyset + h + h = h \neq 1 + h \quad \{$$

$$M_x = 0 \quad \text{exact}$$

$$M_y = 0 \quad \text{exact}$$

$$0 = N_x$$

$$1 + x = N_y$$

$$0 = h \neq (1 + h) + x \neq (1 + x) -$$

$$x \neq (1 + x) \neq h \neq (1 + h)$$

$$\frac{dx}{dy} = \frac{1}{1+x}$$

(16)

$$\text{Ans: } y^2 + y - x^2 - x = 0$$

You may leave your answer in implicit form, if you wish.

$$y'(x) = \frac{2y+1}{2x+1}, \quad y(1) = 1$$

7. (10 pts.) Solve the initial value problem

$$y_1(t) = a$$

$$y_1(t) = 2t$$

$$y_1(t) = t^2$$

$$y''(t) + y(t) = t^2 + 2$$

Check:

$$A_3 = 0$$

$$a + A_3 = a$$

$$2A_1 + A_3 = a$$

$$A_2 = 0$$

$$A_1 = 1$$

$$y''(t) = 2A_1$$

$$(2A_1 + A_3) + A_2(t) + A_1t^2 = t^2 + 2$$

$$y(t) = 2A_1t + A_2$$

$$y(t) = A_1t^2 + A_2t + A_3$$

Ans.: $y(t) = t^2$

10

$$y''(t) + y(t) = t^2 + 2$$

6. (10 pts.) Find a particular solution of the diff. eq.

5. (20 pts.) Set-up, but do not compute, a template for a particular solution of the diff.eq.

(20)

$$y^{(4)}(t) + y'(t) = t^2 + t^2 e^{-t} + \cos t$$

Ans.: A template is (using A_1, A_2, A_3, \dots , etc.) $y(t) = (A_1 t^3 + A_2 t^2 + A_3 t) + t(A_4 t^2 + A_5 t + A_6) e^{-t}$
 $+ A_7 \cos t + A_8 \sin t$

$$r^4 + r^2 = 0$$

$$t^2:$$

$$(A_1 t^3 + A_2 t^2 + A_3 t) t$$

$$r(r^3 + 1) = 0$$

$$t^2 e^{-t}:$$

$$t(A_4 t^2 + A_5 t + A_6) e^{-t}$$

$$r(r+1)(r^2 - r + 1)$$

$$\cos t:$$

$$A_7 \cos t + A_8 \sin t$$

$$r=0, -1, \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Check:

$$t^2: \quad y(t) = A_1 t^3 + A_2 t^2 + A_3 t$$

$$t^2 e^{-t}: \quad y(t) = (A_4 t^3 + A_5 t^2 + A_6 t) e^{-t}$$

$$y'(t) = 3A_1 t^2 + 2A_2 t + A_3$$

$$y'(t) = (3A_4 t^2 + 2A_5 t + A_6) e^{-t} \Rightarrow (A_4 t^3 + A_5 t^2 + A_6 t) e^{-t}$$

$$y''(t) = 6A_1 t + 2A_2$$

$$y''(t) =$$

$$y'''(t) = 6A_1$$

$$y^{(4)}(t) = 0$$

cost:

$$y(t) = A_7 \cos t + A_8 \sin t$$

$$y'(t) = -A_7 \sin t + A_8 \cos t$$

$$y''(t) = -A_7 \cos t - A_8 \sin t$$

$$y'''(t) = -A_7 \sin t - A_8 \cos t$$

$$y^{(4)}(t) = A_7 \cos t + A_8 \sin t$$

4. (20 pts.) Use any (correct) method to solve the initial value system

$$\mathbf{x}'(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}(t), \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Ans.: $\mathbf{x}(t) = \begin{pmatrix} 3e^t + 3te^t \\ e^t \\ e^t \\ e^t \end{pmatrix}$

(20)

$$x_1'(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$$

$$x_2'(t) = x_2(t)$$

$$x_3'(t) = x_3(t)$$

$$x_4'(t) = x_4(t)$$

$$x_1'(t) - x_1(t) = e^t + e^t + e^t$$

$$x_1'(t) - x_1(t) = 3e^t$$

$$x_1(t) = \frac{\int e^{5-1dt} \cdot 3e^t dt}{e^{5-1dt}}$$

$$= \frac{\int e^{-t} \cdot 3e^t dt}{e^{-t}}$$

$$= \frac{\int 3dt}{e^{-t}} = \frac{3t + C}{e^{-t}}$$

$$(3t + C)e^{-t}$$

$$x_1(0) = (3 \cdot 0 + C)e^0 = 3$$

$$C = 3$$

$$x_1(t) = 3te^{-t} + 3e^{-t}$$

Check:

$$x_1(0) = \begin{pmatrix} 3e^0 + 3 \cdot 0e^0 \\ e^0 \\ e^0 \\ e^0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \checkmark$$

$$x_2(0) = Ce^0 = 1 \rightarrow C = 1 \rightarrow x_2(t) = e^t$$

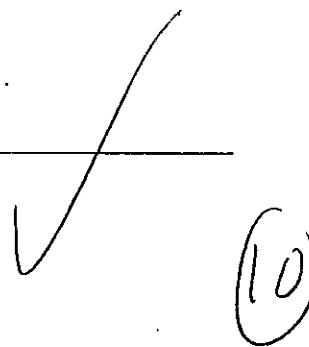
$$x_3(0) = Ce^0 = 1 \rightarrow C = 1 \rightarrow x_3(t) = e^t$$

$$x_4(0) = Ce^0 = 1 \rightarrow C = 1 \rightarrow x_4(t) = e^t$$

3. (10 pts.) Solve the initial value problem

$$y''(t) + y(t) = t ; \quad y(0) = 0 , \quad y'(0) = 2$$

Ans.: $y(t) = \sin t + t$



(10)

Homogeneous:

$$r^2 + 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-0 \pm \sqrt{0-4}}{2}$$

$$= \pm i$$

$$Y_h = C_1 \cos t + C_2 \sin t$$

Particular:

$$y(t) = At + B$$

$$0 + At + B = t$$

$$y'(t) = A$$

$$At + B' = t$$

$$y''(t) = 0$$

$$B = 0$$

$$At = t$$

$$A = 1$$

$$Y_p = t$$

IVP:

$$y(t) = C_1 \cos t + C_2 \sin t + t$$

$$y'(t) = -C_1 \sin t + C_2 \cos t + 1$$

$$y(0) = C_1 = 0$$

$$y'(0) = C_2 + 1 = 2$$

$$C_2 = 1$$

$$y(t) = \sin t + t$$

Check:

$$y(t) = \sin t + t$$

$$y''(t) + y(t) = t$$

$$y'(t) = \cos t + 1$$

$$-\sin t + \sin t + t = t$$

$$y''(t) = -\sin t$$

$$t = t \quad \checkmark$$

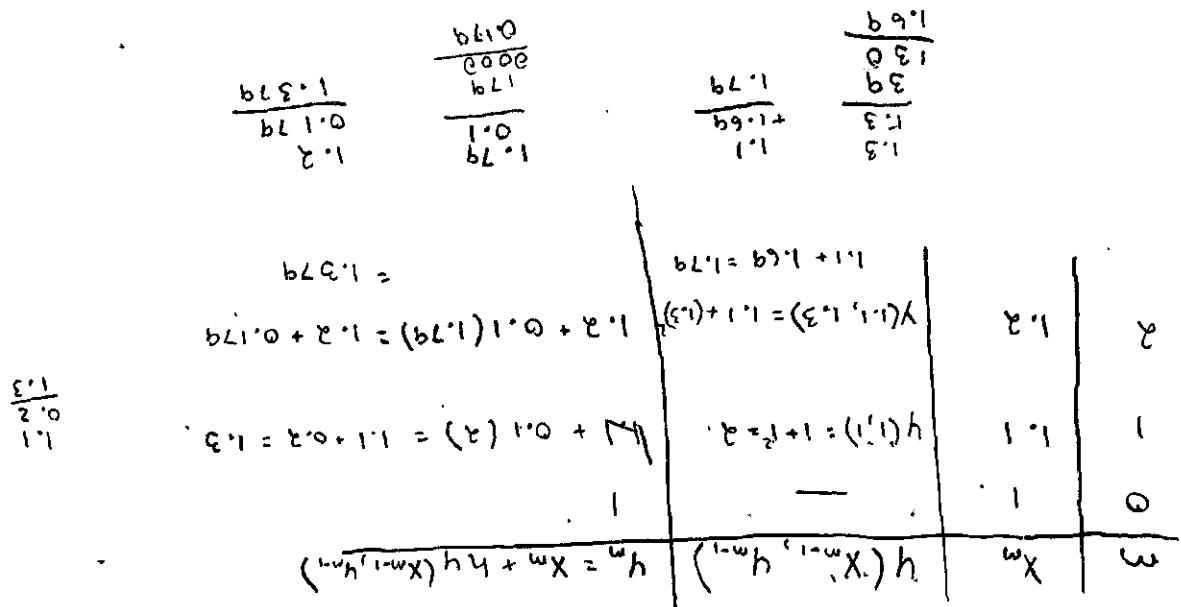
$$y(0) = \sin(0) + 0 = 0$$

$$0 = 0 \quad \checkmark$$

$$y'(0) = \cos(0) + 1 = 2$$

$$1 + 1 = 2$$

$$2 = 2 \quad \checkmark$$



Ans: $y(1.2)$ is approximately equal to: 1.379

using mesh-size $h = 0.1$.

$$f(x) = x + y(x)^2, \quad y(1) = 1,$$

2. (20 pts.) Use the (Simple) Euler method to find an approximate value for $y(1.2)$ if $y(x)$ is the solution of the initial value problem ode

1. (20 pts altogether)

(i) (13 points) Seek a power series solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ to the differential equation

$$y'(x) - (1+2x)y(x) = 0$$

and set-up a recurrence for the coefficients a_n .

(ii) (7 points) Using the recurrence that you found, find the coefficients of x, x^2, x^3, x^4 (i.e. a_1, a_2, a_3, a_4) in the above power series expansion of the general solution $y(x)$. You should leave your answers in terms of a_0 .

Ans.: Recurrence is: $a_{n+1} = \frac{a_n + 2a_{n-1}}{n+1}$

$$a_1 = a_0, \quad a_2 = \frac{3a_0}{2}, \quad a_3 = \frac{7a_0}{6}, \quad a_4 = \frac{25a_0}{24}$$

$$\begin{aligned} n = 4 \rightarrow a_5 &= \frac{a_4 + 2a_3}{5} \\ &= \frac{\frac{25a_0}{24} + \frac{14a_0}{6}}{5} \\ &= \frac{25a_0 + 14a_0}{120} \\ &= \frac{81a_0}{120} x^4 \end{aligned}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$n=1 \rightarrow a_2 = \frac{a_1 + 2a_0}{2} = \frac{a_0 + 2a_0}{2} = \frac{3a_0}{2}$$

$$n=2 \rightarrow a_3 = \frac{a_2 + 2a_1}{3} = \frac{\frac{3a_0}{2} + 2a_0}{3} = \frac{7a_0}{6}$$

$$= \frac{7a_0}{6} = \frac{7a_0}{6} x^2$$

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - (1+2x) \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} 2a_n x^{n+1} = 0$$

$$\begin{matrix} n = 1 \\ m = n-1 \\ n = m+1 \end{matrix} \quad \begin{matrix} n = 0 \\ m = n+1 \\ n = m-1 \end{matrix}$$

$$\sum_{m=0}^{\infty} (m+1) a_{m+1} x^m - \sum_{n=0}^{\infty} a_n x^n - \sum_{m=1}^{\infty} 2a_{m-1} x^m = 0$$

$$\sum_{m=0}^{\infty} (n+1) a_{m+1} x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0$$

$$(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n - a_0 x^0 - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0$$

$$a_1 - a_0 + \sum_{n=1}^{\infty} ((n+1)a_{n+1} - a_n - 2a_{n-1}) x^n = 0$$

$$a_1 - a_0 = 0$$

$$(n+1)a_{n+1} - a_n - 2a_{n-1} = 0$$

$$a_1 = a_0$$

$$a_{n+1} = \frac{a_n + 2a_{n-1}}{n+1}$$

$$n=3 \rightarrow a_4 = \frac{a_3 + 2a_2}{4}$$

$$= \frac{7a_0}{6} + 2 \left(\frac{3a_0}{2} \right) \frac{4}{4}$$

$$= \frac{7a_0}{6} + \frac{3a_0}{2} = \frac{7a_0 + 18a_0}{6} = \frac{25a_0}{6} = \frac{25a_0}{4} = \frac{25a_0}{24} x^4$$

You must check every answer (when applicable), and show your checking!

Reminder: Abel's theorem says that if $y_1(t), y_2(t)$ are *any* two solutions of the linear differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ then their Wronskian, $W(y_1(t), y_2(t))$ equals $ce^{-\int p(t) dt}$, where c is an unknown constant.

Reminder: $r^3 + 1 = (r + 1)(r^2 - r + 1)$.

Reminder: To find the general solution of a first-order linear differential equation $y'(t) + p(t)y(t) = q(t)$, first find the integrating factor, $I(t) = e^{\int p(t) dt}$, and then $y(t) = \frac{\int I(t)q(t) dt}{I(t)}$.

$$e^{iz} = \cos z + i \sin z$$

left a.12 back a.15

NAME: (print!) RUCHI PATEL Section 01

E-Mail address: RJP258@SCARLETMAIL.EDU

SSC(circle) • None • Only SCC1 • Only SCC2 • SCC1 and SCC2

MATH 244 (1-3), Dr. Z., FINAL EXAM, Monday, Dec. 19, 2016, 8:00-11:00am, SEC-117

No Calculators! No Cheatsheets! Write the final answer to each problem in the space provided.

Do not write below this line (office use only)

1. 20 (out of 20)

2. 10 (out of 20)

3. 10 (out of 10)

4. 20 (out of 20)

5. 20 (out of 20)

6. 10 (out of 10)

7. 10 (out of 10)

8. 10 (out of 10)

9. 10 (out of 10)

10. 10 (out of 10)

11. 10 (out of 10)

12. 10 (out of 10)

13. 10 (out of 10)

14. 10 (out of 10)

15. 10 (out of 10)

16. 10 (out of 10)

total: 200 (out of 200)

200