By Doron Zeilberger

The general form of a **homogeneous second order linear diff.eq.** is

\[ a(t) y''(t) + b(t) y'(t) + c(t) y(t) = 0 \]

where, in general, the **coefficients**, \(a(t), b(t), c(t)\) are **functions** of \(t\) (not just **constants**).

If the right side is **not** zero, then we have an **inhomogeneous second order linear diff.eq.** whose format is:

\[ a(t) y''(t) + b(t) y'(t) + c(t) y(t) = d(t) \]

where \(d(t)\) is yet another function of \(t\).

By dividing by \(a(t)\), and putting \(p(t) = \frac{b(t)}{a(t)}, q(t) = \frac{c(t)}{a(t)}, g(t) = \frac{d(t)}{a(t)}\), it may be written

\[ y''(t) + p(t) y'(t) + q(t) y(t) = g(t) \]

where \(p(t), q(t), \) and \(g(t)\) are other functions of \(t\).

**Existence and Uniqueness Theorem for Second-Order Linear Diff.eqs.**

If the coefficient-functions \(p(t), q(t), g(t)\) are **continuous** (i.e. “well-behaved”, they have no breaks and do not blow up) in an open interval \(I\) containing the number \(t_0\), then we are **guaranteed** that there exists a **unique** solution \(y(t)\) to the initial value diff.eq.

\[ y''(t) + p(t) y'(t) + q(t) y(t) = g(t) \quad \text{for any numbers } y_0, z_0, \text{ that makes sense through the open interval } I. \]

In other words, if the functions \(p(t), q(t), g(t)\) do not blow up, and have no surprises in \(I\), and want a function \(y(t)\) that satisfies the diff.eq. and its value at \(t_0\) is \(y_0\), and the value of its rate-of-change at \(t_0\) is \(z_0\), then you are promised that there is such a function. Not only that, there is only one such function!

**Problem 9.1:** For each of the following diff.eq. initial value problems, and intervals, decide whether the theorem promises you that there is a unique solution.

- **a.** \((t^2 + 1) y''(t) + \sin t y'(t) + (t^3 + 1) y(t) = e^t \quad y(0) = 1 \quad y'(0) = 3 \quad -10 < t < 10\)
- **b.** \(t^2 y''(t) + \sin t y'(t) + (t^3 + 1) y(t) = e^t \quad y(0) = 1 \quad y'(0) = 3 \quad -10 < t < 10\)
- **c.** \(y''(t) + \tan t y'(t) + (t^3 + 1) y(t) = e^{t^2} \quad y(0) = 1 \quad y'(0) = 3 \quad 0 < t < \frac{2\pi}{3}\)
- **d.** \((t - 5)(t - 3) y''(t) + y'(t) + y(t) = e^{t^2} \quad y(0) = 1 \quad y'(0) = 3 \quad -2 < t < 2\)
Solution to 9.1a: We first divide by the coefficient of $y''(t)$, $t^2 + 1$, getting the equivalent diff.eq. (initial value problem)

\[ a. \ y''(t) + \frac{\sin t}{t^2 + 1} y'(t) + \frac{t^2 + 1}{t^2 + 1} y(t) = \frac{t^3}{t^2 + 1} , \quad y(0) = 1 , \quad y'(0) = 3 ; \quad -10 < t < 10 \]

So $p(t) = \frac{\sin t}{t^2 + 1}$, $q(t) = \frac{t^2 + 1}{t^2 + 1}$, and $g(t) = \frac{t^3}{t^2 + 1}$. The numerators are all nice (continuous) functions, and the denominator in these, $t^2 + 1$ never vanishes in the interval $(-10, 10)$, so everything is nice, and the theorem promises a unique solution.

Solution to 9.1b: We first divide by the coefficient of $y''(t)$, $t^2$, getting the equivalent diff.eq. (initial value problem)

\[ b. \ y''(t) + \frac{\sin t}{t^2} y'(t) + \frac{t^3 + 1}{t^2} y(t) = \frac{t}{t^2} , \quad y(0) = 1 , \quad y'(0) = 3 ; \quad -10 < t < 10 \]

So $p(t) = \frac{\sin t}{t^2}$, $q(t) = \frac{t^3 + 1}{t^2}$, and $g(t) = \frac{t}{t^2}$. The numerators are all nice (continuous) functions, but the denominator in these, $t^2$ vanishes at $t = 0$, and $t = 0$ happens to lie in the given interval, so (at least one of, but in this case all of) $p(t), q(t), g(t)$ blow up, so are not continuous, and there is no guarantee.

Solution of 9.1c: $\tan t$ blows up at $t = \frac{\pi}{2}$ and since it lies in the given interval $0 < t < \frac{2\pi}{3}$, it is not continuous and there is no guarantee.

Solution of 9.1d: After dividing by $(t - 3)(t - 5)$ we see that the coefficient functions blow-up at $t = 3$ and $t = 5$. But neither trouble spots lie in the interval $-2 < t < 2$ so we are safe and we are guaranteed a unique solution.

Ans. to 9.1: a and d are guaranteed to have a unique solution, but b and c are not.

Problem 9.2: Find the largest open interval for which the solution to

\[ (t - 1)(t + 3) y''(t) + \sin ty'(t) + \cos ty(t) = t , \quad y(0) = 1 , \quad y'(0) = 3 \]

has a unique solution.

Solution to 9.2: After dividing by the coefficient of $y''(t)$, $(t - 1)(t + 3)$, we get the initial value problem

\[ y''(t) + \frac{\sin t}{(t - 1)(t + 3)} y'(t) + \frac{\cos t}{(t - 1)(t + 3)} y(t) = \frac{t}{(t - 1)(t + 3)} , \quad y(0) = 1 , \quad y'(0) = 3 \]

The coefficients blow-up at $t = -3$ and $t = 1$. Our interval should contain $t_0 = 0$, so the largest interval is $-3 < t < 1$.

Ans. to 9.2: The largest interval is $-3 < t < 1$.

We already know about the amazing Principle of Superposition (only valid for homogeneous diff.eqs.!)
If \( y_1(t) \) and \( y_2(t) \) are two solutions of the second-order homog. linear diff.eq.

\[
y''(t) + p(t)y'(t) + q(t)y(t) = 0,
\]

then we can come up with (doubly!) infinite family of functions that also satisfy the same diff.e.q

\[
c_1y_1(t) + c_2y_2(t)
\]

where \( c_1, c_2 \) are arbitrary constants free to range from \(-\infty\) to \(\infty\).

Usually, every solution is of that form. To find out whether this is the case we need the important concept of Wronskian.

**Important Definition (The Wronskian):** The Wronskian of two functions \( f(t), g(t) \) is the brand-new function

\[
W(f(t), g(t)) = f(t)g'(t) - f'(t)g(t)
\]

**Problem 9.3:** Find the Wronskian, \( W(f(t), g(t)) \) of the following pair of functions.

a: \( f(t) = t^3, \quad g(t) = t^2 \)
b: \( f(t) = e^t, \quad g(t) = \sin t \)
c: \( f(t) = \sin t, \quad g(t) = \cos t \)

**Solution to 9.3:**

a.: \( W(t^3, t^2) = t^3(t^2)' - (t^3)'(t^2) = t^3(2t) - (3t^2)(t^2) = 2t^4 - 3t^4 = -t^4 \)
b.: \( W(e^t, \sin t) = e^t(\sin t)' - (e^t)'(\sin t) = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t) \)
c.: \( W(\sin t, \cos t) = \sin t(\cos t)' - (\sin t)'(\cos t) = -\sin^2 t - \cos^2 t = -1 \)

**Important Theorem:** Suppose that \( y_1(t) \) and \( y_2(t) \) are two specific solutions of the second-order linear diff.eq.

\[
y''(t) + p(t)y'(t) + q(t)y(t) = 0,
\]

Then every other solution can be written as \( c_1y_1(t) + c_2y_2(t) \) for some constants \( c_1, c_2 \) if and only if the Wronskian, \( W(y_1, y_2) \) is not identically zero.

Note: it is OK if the Wronskian has some zero-point(s). For example if \( y_1(t) = t, \ y_2(t) = t^2 \) then \( W(t, t^2) = t(t^2)' - (t)'(t^2) = t^2 \) and it is 0 when \( t = 0 \), but \( t \) and \( t^2 \) are still an OK fundamental set. But if \( y_1 = t^2, \ y_2 = 2t^2 \) then \( W(t^2, 2t^2) = 0 \) (always!), so these are not OK.

Most diff.eq. are impossible to solve (or very hard), so given a linear second-order diff.eq.

\[
y''(t) + p(t)y'(t) + q(t)y(t) = 0,
\]

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and you want to find the Wronskian of any two of its solutions \( y_1(t), y_2(t) \), then it is either very hard, or impossible, to compute the Wronskian of \( y_1(t), y_2(t) \) directly. But thanks to Nils Abel, we don’t! We can figure it out (up to a constant multiple \( c \)) from the diff.eq. itself \textbf{without} bothering to solve it!

**Abel’s Theorem**

The Wronskian of any two solutions \( y_1(t), y_2(t) \) of the above linear homog. diff.eq. is always given by

\[
c \exp \left[ - \int p(t) \, dt \right],
\]

where \( c \) is a constant.

\textbf{Note:} If \( y_1(t) \) and \( y_2(t) \) are constant multiples of each other then \( c \) happens to be 0.

**Problem 9.4:** Find the Wronskian (up to a constant in front) of any two solutions of the following diff.eq.

\[
y''(t) + t^2 y'(t) + e^t y(t) = 0.
\]

\textbf{Sol. to 9.4:} Here \( p(t) = t^2 \) (note: \( q(t) \) is irrelevant for this problem!), we have

\[
W(y_1, y_2) = c \exp \left[ - \int t^2 \, dt \right] = c \exp \left[ - \frac{t^3}{3} \right] = ce^{-t^3/3}.
\]

\textbf{Ans. to 9.4:} The Wronskian is \( ce^{-t^3/3} \) for some constant \( c \).