Dr. Z.’s Calc4 Lecture 8 Handout: 
Homogeneous Second-Order Differential Equations With Constant Coefficients

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The general form of a *general* (usually non-linear) *second-order* diff.eq. is

\[ y''(t) = F(t, y(t), y'(t)) \]

Most diff. eq. can’t be solved “analytically”, i.e. via a closed-form expression featuring polynomials and the familiar exponential and trig functions, and one can only find numerical approximations. But for **special classes** of 2nd order diff.eqs. there are **special methods**.

An important class (still difficult) is the family of **linear second-order differential equations**. Its format is

\[ P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = S(t) \]

where \( P(t), Q(t), R(t), S(t) \) are some functions of \( t \). For example, the following are linear diff.eqs.

\[ t \ y''(t) + \sin t \ y'(t) + e^t y(t) = t^3 \]

but the following is **not** linear

\[ t \ y''(t) + \sin t \ y'(t) + e^t y(t)^2 = t^3 \]

since \( y(t) \) appears squared. Neither is

\[ t \ y''(t) + \sin t \ y'(t)y(t) = t^3 \]

since \( y(t) \) and \( y'(t) \) are multiplied by each other. To qualify as **linear** each of \( y''(t), y'(t) \) and \( y(t) \) must be **alone**.

If the right-side, \( S(t) \) happens to be 0, i.e. we have a linear 2nd order diff.eq. of the format

\[ P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0 \]

then such a linear diff.eq. is called **homogeneous**.

Homogeneous diff.eqs. have two **amazing** properties.

**First amazing property:** If \( \phi(t) \) is a solution of a **homogeneous** linear (2nd-order) diff. eq. \( P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0 \), then so is any **constant** multiple, \( c\phi(t) \) (\( c \) a number).

**Second amazing property:** If \( \phi_1(t) \) and \( \phi_2(t) \) are both solutions of a **homogeneous** linear (2nd-order) diff. eq. \( P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0 \), then so is their sum: \( \phi_1(t) + \phi_2(t) \).

The proofs are easy (you do it!).

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Even for the special case of homog. 2nd-order linear diff. eqs. it is usually not possible to solve them (of course you can declare the solution to be a new function, and name it after yourself). But if the coefficients $P(t), Q(t), R(t)$ happen to be constant functions (do not depend on $t$), then one can always find explicit solutions.

A general **homogeneous 2nd-order linear diff.eq. with constant coefficients** has the format

$$ay''(t) + by'(t) + cy(t) = 0$$

where $a, b, c$ are (fixed!) numbers. You try your luck with a solution of the form

$$y(t) = e^{rt}$$

and plug it into the diff.eq. We have $y(t) = e^{rt}$, $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$

and get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

Factoring out $e^{rt}$:

$$(ar^2 + br + c)e^{rt} = 0$$

Since $e^{rt}$ is never zero, we can divide by it (or if you wish, multiply by $e^{-rt}$) and get

$$ar^2 + br + c = 0$$

This is no longer a differential equation, but rather a simple (quadratic) **algebraic equation**.

If it has two distinct real roots (we will talk about multiple and complex roots later), let’s call them $r_1, r_2$, then it means that we found two independent solutions $e^{r_1t}$ and $e^{r_2t}$. By the first amazing property any constant multiples (with [usually] different constants) $c_1e^{r_1t}$ and $c_2e^{r_2t}$ are also solutions. By the second amazing property for homogeneous linear diff.eqs so it the sum, so

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

are all solutions of the diff.eq. $ay''(t) + by'(t) + cy(t) = 0$. It is possible to prove that every solution is of that form, so we have
Important Theorem

The general solution of the homogeneous linear second-order differential equation with constant coefficients

\[ ay''(t) + by'(t) + cy(t) = 0 , \]

is

\[ y(t) = C_1e^{r_1t} + C_2e^{r_2t} , \]

where \( r_1, r_2 \) are the two distinct roots of the quadratic equation (assuming that \( b^2 - 4ac > 0 \))

\[ ar^2 + br + c = 0 , \]

and \( C_1, C_2 \) are arbitrary constants.

Note: The quadratic equation \( ar^2 + br + c = 0 \) is called the characteristic equation.

Problem 8.1 Find the general solution of the following diff. eq.

\[ y'' - 3y' + 2y = 0 . \]

Step 1: Write down the characteristic equation by replacing \( y'' \) by \( r^2 \), \( y' \) by \( r \) and \( y \) by 1

\[ r^2 - 3r + 2 = 0 . \]

Step 2: Solve the quadratic equation, either using \( r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), or, if possible, by factorization.

\[ (r - 1)(r - 2) = 0 , \]

so we have two distinct roots, \( r_1 = 1, r_2 = 2 \).

Step 3: Write down the general solution

\[ y(t) = C_1e^{r_1t} + C_2e^{r_2t} . \]

In this problem the two roots are \( r_1 = 1, r_2 = 2 \), so we have

\[ y(t) = C_1e^t + C_2e^{2t} . \]

Ans. to 8.1: \( y(t) = C_1e^t + C_2e^{2t} \).

Problem 8.2 Find the general solution of the following diff. eq.

\[ y'' - 3y' + y = 0 . \]
Step 1: Write down the characteristic equation by replacing $y''$ by $r^2$, $y'$ by $r$ and $y$ by 1

$$r^2 - 3r + 1 = 0$$

Step 2: Solve the quadratic equation, either using $-b \pm \sqrt{b^2 - 4ac}/2a$, or, if possible, by factorization. Now it is not possible to factorize, so we have

$$r_1, r_2 = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}$$

so we have two distinct roots, $r_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}, r_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$.

Step 3: Write down the general solution

$$y(t) = C_1 e^{\left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)t} + C_2 e^{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)t}$$

Ans. to 8.2: $y(t) = e^{\frac{3}{2}t}(C_1 e^{\frac{\sqrt{5}}{2}t} + C_2 e^{-\frac{\sqrt{5}}{2}t})$.

The initial value problems for second-order diff. eqs. have two data $y(0), y'(0)$ (or more generally $y(t_0), y'(t_0)$).

Problem 8.3 Find the solution of the following initial value 2nd-order diff. eq.

$$y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3$$

Step 1-3: Find the general solution, like in Problem 8.1.

$$y(t) = C_1 e^t + C_2 e^{2t}$$

Step 4: Find $y'(t)$

$$y'(t) = C_1 e^t + 2C_2 e^{2t}$$

and plug-in $t = t_0$ into the general $y(t)$ and $y'(t)$.

$$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$$

$$y'(0) = C_1 e^0 + 2C_2 e^0 = C_1 + 2C_2$$

Step 5: Implement the initial conditions:

$$2 = C_1 + C_2$$

$$3 = C_1 + 2C_2$$

Step 6: Solve the system of two equations and two unknowns getting $C_1, C_2$, certain specific numbers. Subtracting the second eq. from the first gives $C_2 = 1$, plugging into the first (or the second) gives $C_1 = 1$. 
Step 7: Go back to the general solution above and substitute the $C_1, C_2$.

$$y(t) = 1 \cdot e^t + 1 \cdot e^{2t} = e^t + e^{2t}.$$  

Ans. to 8.3: $y(t) = e^t + e^{2t}$.

Problem 8.4 Find the solution of the following initial value diff. eq.

$$y'' - y = 0 \quad , \quad y(-2) = 1 \quad , \quad y'(-2) = -1.$$  

Steps 1-3: We first find the general solution, featuring $C_1, C_2$. (i) $r^2 - 1 = 0$ (ii) $r = -1, r = 1$ (iii) $y(t) = C_1 e^{-t} + C_2 e^t$

Step 4: Find $y'(t)$

$$y'(t) = -C_1 e^{-t} + C_2 e^t$$

and plug-in $t = t_0$ into the general $y(t)$ and $y'(t)$.

$$y(-2) = C_1 e^2 + C_2 e^{-2}$$

$$y'(-2) = -C_1 e^2 + C_2 e^{-2}$$

Step 5: Implement the initial conditions:

$$1 = C_1 e^2 + C_2 e^{-2}$$

$$-1 = -C_1 e^2 + C_2 e^{-2}$$

Step 6: Solve the system of two equations and two unknowns getting $C_1, C_2$, certain specific numbers.

Adding them we get: $2C_2 e^{-2} = 0$ so $C_2 = 0$. Subtracting (or plugging-in) gives $2 = 2C_1 e^2$ so $C_1 = e^{-2}$ and $C_2 = 0$.

Step 7: Go back to the general solution above and substitute the $C_1, C_2$.

$$y(t) = C_1 e^{-t} + C_2 e^t = e^{-t} + 0e^t = e^{-t}.$$  

Ans. to 8.4: $y(t) = e^{-t}$.  
