An autonomous first-order diff. eq. is a special case of a separable equation that has the form

\[ y'(t) = f(y) \]

where the function on the right side only depends on \( y \). Of course we can solve it, at least in principle, by the method of separation of variables

\[ \frac{dy}{f(y)} = dt \]

and integrating both sides

\[ \int \frac{dy}{f(y)} = t + C \]

and if we (or Maple) can do the integration, we have at least an implicit solution.

But we can also solve them quantitatively using a graphical method. We first find the equilibrium values of \( y \) by solving the algebraic equation

\[ f(y) = 0 \]

getting either no solutions (e.g. \( f(y) = y^2 + 1 \)), one solution (e.g. \( f(y) = y - 4 \), \( f(y) = (y - 3)(y^2 + 9) \)), two solutions (e.g. \( f(y) = (y^2 - 1) \), \( f(y) = (y^2 - 1)(y^2 + 4) \)), or even more (e.g. \( f(y) = y(y - 1)(y - 2) \)).

For each solution, \( y_0 \) of \( f(y) = 0 \), we know automatically that the horizontal line \( y(t) = y_0 \) is a solution since of course \( y'(t) = 0 \) for such horizontal lines.

Now, for each such equilibrium solution \( y = y_0 \) there are two possibilities as far as starting a little above it

**stable from above**: Any solution that starts a little above \( y = y_0 \) eventually converges to \( y = y_0 \), this happens when \( f(y_0 + \text{tiny}) < 0 \), so the slope is negative and the curve goes down.

**unstable from above**: Any solution that starts a little above \( y = y_0 \) eventually goes away from \( y = y_0 \) this happens when \( f(y_0 + \text{tiny}) > 0 \), so the slope is positive, and the curve goes away from our horizontal line.

Also, for each such equilibrium solution \( y = y_0 \) there are two possibilities as far as starting a little below it
**stable from below:** Any solution that starts a little below \( y = y_0 \) eventually converges to \( y = y_0 \), this happens when \( f(y_0 - \text{tiny}) > 0 \), so the curve goes up eventually getting ever so close to \( y = y_0 \).

**unstable from below:** Any solution that starts a little below \( y = y_0 \) eventually goes away from \( y = y_0 \) this happens when \( f(y_0 - \text{tiny}) < 0 \), so the curve goes down and away from our horizontal line \( y = y_0 \).

If an equilibrium solution \( y = y_0 \) is both “stable from above” and “stable from below” it is called **asymptotically stable**.

If an equilibrium solution \( y = y_0 \) is both “unstable from above” and “unstable from below” it is called **asymptotically unstable**.

If an equilibrium solution \( y = y_0 \) is “unstable from above” but “stable from below”, or vice versa, it is called **asymptotically semistable**.

The most famous autonomous diff.eq. is the **Logistic Eq.** that has the form

\[
y'(t) = r(1 - \frac{y}{K})y,
\]

for some numbers \( r \) (that can be positive or negative), and \( K \). When \( r > 0 \) the solution \( y = K \) is stable, but when \( r < 0 \) it is unstable.

**Problem 5.1:** For the following diff. eq. determine the critical (equilibrium) solution and decide whether it is asymptotically stable, unstable, or semi-stable.

\[
\frac{dy}{dt} = y - 4.
\]

**Solution of 5.1:** Here \( f(y) = y - 4 \). Solving it we get that there is only one equilibrium solution, \( y = 4 \).

Now \( f(4.000001) = 4.000001 - 4 = 0.00001 > 0 \) so the curve goes up, **away** from \( y = 4 \), and it is unstable from above.

Also \( f(3.999999) = 3.999999 - 4 = -0.000001 < 0 \) so the curve goes down, **away** from \( y = 4 \), and it is unstable from below.

So \( y = 4 \) is **asymptotically unstable**.

**Ans. to 5.1:** There is only one equilibrium solution, \( y = 4 \), and it is asymptotically unstable.

**Problem 5.2:** For the following diff. eq. determine the critical (equilibrium) solution and decide whether it is asymptotically stable, unstable, or semi-stable.

\[
\frac{dy}{dt} = 4 - y
\]
Solution of 5.2: Here \( f(y) = 4 - y \). Solving it we get that there is only one equilibrium solution, \( y = 4 \).

Now \( f(4.000001) = 4 - 4.000001 = -0.00001 < 0 \) so the curve goes down, towards \( y = 4 \), and it is stable from above.

Also \( f(3.999999) = 4 - 3.999999 = 0.000001 > 0 \) so the curve goes up, towards \( y = 4 \), and it is stable from below.

So \( y = 4 \) is asymptotically stable.

Ans. to 5.2: There is only one equilibrium solution, \( y = 4 \), and it is asymptotically stable.

Problem 5.3: For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

\[
\frac{dy}{dt} = (y + 2)(y - 1) .
\]

Solution of 5.3: Here \( f(y) = (y + 2)(y - 1) \). Solving it we get that there are two equilibrium solution, \( y = -2 \) and \( y = 1 \).

For \( y = -2 \) and \( tiny = 0.01 \) we have

\[
f(-1.99) = (-1.99 + 2)(-1.99 - 1) < 0
\]

so it is stable from above. Also, going below \( y = -2 \), we have

\[
f(-2.01) = (-2.01 + 2)(-2.01 - 1) = (-0.01)(-3.01) > 0
\]

so it is stable from below. So the equilibrium solution \( y = -2 \) is asymptotically stable.

For \( y = 1 \) and \( tiny = 0.01 \) we have

\[
f(1.01) = (1.01 + 2)(1.01 - 1) > 0
\]

so it is unstable from above. Going a little bit below \( y = 1 \),

\[
f(0.99) = (0.99 + 2)(0.99 - 1) < 0
\]

so it is unstable from below. So the equilibrium solution \( y = 1 \) is asymptotically unstable.

Problem 5.4: For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether it is asymptotically stable, unstable, or semi-stable.

\[
\frac{dy}{dt} = 2(y - 2)^2 .
\]
**Solution of 5.4:** Here \( f(y) = 2(y - 2)^2 \). Solving it we get that there is only one equilibrium solution, \( y = 2 \).

Going a tiny bit above \( y = 2 \), say, \( y = 2.01 \), we have

\[
f(2.01) = 2(2.01 - 2)^2 = 2(0.01)^2 > 0 ,
\]

so it it unstable from above.

Going a tiny bit below \( y = 2 \), say, \( y = 1.99 \), we have

\[
f(1.99) = 2(1.99 - 2)^2 = 2(-0.01)^2 > 0 ,
\]

so it it stable from below.

Since \( y = 2 \) is unstable from above but stable from below, it is **asymptotically semistable**.

**Ans. to 5.4:** There is only one equilibrium solution, \( y = 2 \), it is asymptotically semistable.

**Problem 5.5:** For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

\[
\frac{dy}{dt} = y(y - 2)(y - 4) .
\]

**Solution of 5.5:** Here \( f(y) = y(y - 2)(y - 4) \). Solving it we get that there are three equilibrium solutions, \( y = 0 \), \( y = 2 \), and \( y = 4 \).

For \( y = 0 \), going a tiny bit above \( y = 0 \), say, \( y = 0.01 \), we have

\[
f(0.01) = (0.01)(0.01 - 2)(0.01 - 4) > 0 ,
\]

so the curve goes up and it is unstable from above.

Still for \( y = 0 \), going a tiny bit below \( y = 0 \), say, \( y = -0.01 \), we have

\[
f(-0.01) = (-0.01)(-0.01 - 2)(-0.01 - 4) < 0 ,
\]

so the curve goes down and it is unstable from below.

So the equilibrium solution \( y = 0 \) is **asymptotically unstable**.

For \( y = 2 \), going a tiny bit above \( y = 2 \), say, \( y = 2.01 \), we have

\[
f(2.01) = (2.01)(2.01 - 2)(2.01 - 4) < 0 ,
\]

so the curve goes down and it is stable from above.
Still for $y = 2$, going a tiny bit below $y = 2$, say, $y = 1.99$, we have
\[ f(1.99) = (1.99)(1.99 - 2)(1.99 - 4) > 0 \]
so the curve goes up and it is stable from below.

So the equilibrium solution $y = 2$ is **asymptotically stable**.

For $y = 4$, going a tiny bit above $y = 4$, say, $y = 4.01$, we have
\[ f(4.01) = (4.01)(4.01 - 2)(4.01 - 4) > 0 \]
so the curve goes up and it is unstable from above.

Still for $y = 4$, going a tiny bit below $y = 4$, say, $y = 3.99$, we have
\[ f(3.99) = (3.99)(3.99 - 2)(3.99 - 4) < 0 \]
so the curve goes down and it is unstable from below.

So the equilibrium solution $y = 4$ is **asymptotically unstable**.

**Ans. to 5.5:** There are three equilibrium solutions, $y = 0$, that is asymptotically stable, $y = 2$, that is asymptotically unstable, and $y = 4$ that is asymptotically stable.