Dr. Z.'s Calc4 Lecture 5 Handout: Autonomous Equation

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An **autonomous** first-order diff.eq. is a special case of a separable equation that has the form

$$y'(t) = f(y) \quad ,$$

where the function on the right side **only** depends on y. Of course we can solve it, at least in principle, by the method of separation of variables

$$\frac{dy}{dt} = f(y) \quad ,$$

means

$$\frac{dy}{f(y)} = dt \quad ,$$

and integrating both sides

$$\int \frac{dy}{f(y)}\, dy = t + C \quad ,$$

and if we (or Maple) can do the integration, we have at least an implicit solution.

But we can also solve them *quantitatively* using a graphical method. We first find the **equilibrium** values of y by solving the **algebraic** equation

$$f(y) = 0 \quad ,$$

getting either no solutions (e.g. $f(y) = y^2 + 1$), one solution (e.g. $f(y) = y - 4$, $f(y) = (y - 3)(y^2 + 9)$), two solutions (e.g. $f(y) = (y^2 - 1)$, $f(y) = (y^2 - 1)(y^2 + 4)$), or even more (e.g. $f(y) = y(y - 1)(y - 2)$).

For each solution, y_0 of $f(y) = 0$, we know automatically that the *horizontal line* $y(t) = y_0$ is a solution since of course $y'(t) = 0$ for such horizontal lines.

Now, for each such equilibrium solution $y = y_0$ there are two possibilities as far as *starting a little above it*

stable from above : Any solution that starts a little above $y = y_0$ eventually converges to $y = y_0$, this happens when $f(y_0 + tiny) < 0$, so the slope is **negative** and the curve goes **down**.

unstable from above : Any solution that starts a little above $y = y_0$ eventually goes away from $y = y_0$ this happens when $f(y_0 + tiny) > 0$, so the slope is **positve**, and the curve goes away from our horizontal line.

Also, for each such equilibrium solution $y = y_0$ there are two possibilities as far as *starting a little below it*

stable from below: Any solution that starts a little below $y = y_0$ eventually converges to $y = y_0$, this happens when $f(y_0 - tiny) > 0$, so the curve goes up eventually getting ever so close to $y = y_0$.

unstable from below : Any solution that starts a little below $y = y_0$ eventually goes away from $y = y_0$ this happens when $f(y_0 - tiny) < 0$, so the curve goes down and away from our horizontal line $y = y_0$.

If an equilibrium solution $y = y_0$ is **both** "stable from above" and "stable from below" it is called **asymptotically stable**.

If an equilibrium solution $y = y_0$ is **both** "unstable from above" and "unstable from below" it is called **asymptotically unstable**.

If an equilibrium solution $y = y_0$ is "unstable from above" but "stable from below", or vice versa, it is called **asymptotically semistable**.

The most famous autonomous diff.eq. is the **Logistic Eq.** that has the form

$$y'(t) = r(1 - \frac{y}{K})y \quad ,$$

for some numbers r (that can be positive or negative), and K. When $r > 0$ the solution $y = K$ is stable, but when $r < 0$ it is unstable.

Problem 5.1: For the following diff. eq. determine the critical (equilibrium) solution and decide whether it is asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y - 4 \quad .$$

Solution of 5.1: Here $f(y) = y - 4$. Solving it we get that there is only one equilibrium solution, $y = 4$.

Now $f(4.000001) = 4.000001 - 4 = 0.00001 > 0$ so the curve goes up, **away** from $y = 4$, and it is unstable from above.

Also $f(3.999999) = 3.999999 - 4 = -0.0000001 < 0$ so the curve goes down, **away** from $y = 4$, and it is unstable from below.

So $y = 4$ is **asymptotically unstable**.

Ans. to 5.1: There is only one equilibrium solution, $y = 4$, and it is asymptotically unstable.

Problem 5.2: For the following diff. eq. determine the critical (equilibrium) solution and decide whether it is asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = 4 - y \quad .$$

Solution of 5.2: Here $f(y) = 4 - y$. Solving it we get that there is only one equilibrium solution, $y = 4$.

Now $f(4.000001) = 4 - 4.000001 = -0.00001 < 0$ so the curve goes down, **towards** $y = 4$, and it is stable from above.

Also $f(3.999999) = 4 - 3.999999 = 0.0000001 > 0$ so the curve goes up, **towards** $y = 4$, and it is stable from below.

So $y = 4$ is **asymptotically stable**.

Ans. to 5.2: There is only one equilibrium solution, $y = 4$, and it is asymptotically stable.

Problem 5.3: For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = (y + 2)(y - 1) \quad .$$

Solution of 5.3: Here $f(y) = (y + 2)(y - 1)$. Solving it we get that there are two equilibrium solution, $y = -2$ and $y = 1$.

For $y = -2$ and $tiny = 0.01$ we have

$$f(-1.99) = (-1.99 + 2)(-1.99 - 1) < 0$$

so it is stable from above. Also, going below $y = -2$, we have

$$f(-2.01) = (-2.01 + 2)(-2.01 - 1) = (-0.01)(-3.01) > 0$$

so it is stable from below. So the equilibrium solution $y = -2$ is **asymptotically stable**.

For $y = 1$ and $tiny = 0.01$ we have

$$f(1.01) = (1.01 + 2)(1.01 - 1) > 0$$

so it is unstable from above. Going a little bit below $y = 1$,

$$f(0.99) = (0.99 + 2)(0.99 - 1) < 0$$

so it is unstable from below. So the equilibrium solution $y = 1$ is **asymptotically unstable**.

Problem 5.4: For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether it is asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = 2(y - 2)^2 \quad .$$

Solution of 5.4: Here $f(y) = 2(y-2)^2$. Solving it we get that there is only one equilibrium solution, $y = 2$.

Going a tiny bit above $y = 2$, say, $y = 2.01$, we have

$$f(2.01) = 2(2.01-2)^2 = 2(0.01)^2 > 0 \quad ,$$

so it it unstable from above.

Going a tiny bit below $y = 2$, say, $y = 1.99$, we have

$$f(1.99) = 2(1.99-2)^2 = 2(-0.01)^2 > 0 \quad ,$$

so it it stable from below.

Since $y = 2$ is unstable from above but stable from below, it is **asymptotically semistable**.

Ans. to 5.4: There is only one equilibrium solution, $y = 2$, it is asymptotically semistable.

Problem 5.5: For the following diff. eq. determine the critical (equilibrium) solution(s) and decide whether they are asymptotically stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y(y-2)(y-4) \quad .$$

Solution of 5.5: Here $f(y) = y(y-2)(y-4)$. Solving it we get that there are three equilibrium solutions, $y = 0$. $y = 2$, and $y = 4$.

For $y = 0$, going a tiny bit above $y = 0$, say, $y = 0.01$, we have

$$f(0.01) = (0.01)(0.01-2)(0.01-4) > 0 \quad ,$$

so the curve goes up and it is unstable from above.

Still for $y = 0$, going a tiny bit below $y = 0$, say, $y = -0.01$, we have

$$f(-0.01) = (-0.01)(-0.01-2)(-0.01-4) < 0 \quad ,$$

so the curve goes down and it is unstable from below.

So the equilibrium solution $y = 0$ is **asymptotically unstable**.

For $y = 2$, going a tiny bit above $y = 2$, say, $y = 2.01$, we have

$$f(2.01) = (2.01)(2.01-2)(2.01-4) < 0 \quad ,$$

so the curve goes down and it is stable from above.

Still for $y = 2$, going a tiny bit below $y = 2$, say, $y = 1.99$, we have

$$f(1.99) = (1.99)(1.99 - 2)(1.99 - 4) > 0 \quad ,$$

so the curve goes up and it is stable from below.

So the equilibrium solution $y = 2$ is **asymptotically stable**.

For $y = 4$, going a tiny bit above $y = 4$, say, $y = 4.01$, we have

$$f(4.01) = (4.01)(4.01 - 2)(4.01 - 4) > 0 \quad ,$$

so the curve goes up and it is unstable from above.

Still for $y = 4$, going a tiny bit below $y = 4$, say, $y = 3.99$, we have

$$f(3.99) = (3.99)(3.99 - 2)(3.99 - 4) < 0 \quad ,$$

so the curve goes down and it is unstable from below.

So the equilibrium solution $y = 4$ is **asymptotically unstable**.

Ans. to 5.5: There are three equilibrium solutions, $y = 0$, that is asymptotically stable, $y = 2$, that is asymptotically unstable, and $y = 4$ that is asymptotically stable.