By Doron Zeilberger

The best kind of functions are **polynomials** that have the format

\[ f(x) = a_0 + a_1 x + \ldots + a_n x^n \]

for some finite \( n \), called the **degree** of the polynomial. If we are really lucky, we may find a solution of a diff.eq. that is a polynomial. For example the diff.eq.

\[ y''(x) + xy'(x) - 2y(x) = 0 \]

If someone told you that there is a solution that is a polynomial of degree 2, you can find it, by trying the **template**

\[ y(x) = a_0 + a_1 x + a_2 x^2 \]

featuring **undetermined coefficients** \( a_0, a_1, a_2 \). To find them, you first find expressions, in terms of \( a_0, a_1, a_2 \) for \( y'(x) \) and \( y''(x) \):

\[
\begin{align*}
y'(x) &= a_1 + 2a_2 x \\
y''(x) &= 2a_2 
\end{align*}
\]

You now plug this template into the diff.eq. getting

\[ 2a_2 + x(a_1 + 2a_2 x) - 2(a_0 + a_1 x + a_2 x^2) = 2a_2 + a_1 x + 2a_2 x^2 - 2a_0 - 2a_1 x - 2a_2 x^2 \]

Now you collect coefficients getting

\[ (2a_2 - 2a_0) - a_1 x \]

Since this is **identically zero**, each coefficient must be zero, so we have to solve the system

\[ 2a_2 - 2a_0 = 0 , \quad a_1 = 0 \]

whose solution is \( a_0 = \text{anything} \), \( a_1 = 0 \), \( a_2 = a_0 \), so a solution is \( a_0 + 0 \cdot x + a_0 x^2 = a_0(1 + x^2) \).

So we lucked out and found a polynomial solution, \( f(x) = 1 + x^2 \) and of course (since the diff.eq. is linear and homogeneous) any constant multiple of it.

This was luck! Consider a general **linear homogeneous second-order differential equation**

\[ P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0 \]

For the sake of simplicity, let’s look for solutions near \( x = 0 \) (we can always make a change of variable to move it to any desired point). If \( P(x), Q(x), R(x) \) are nice (often these are polynomials)
and $P(x)$ does not vanish at $x = 0$ (or, in general, at $x = x_0$) we can divide by $P(x)$ getting a simplified diff.eq. of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p(x)$ and $q(x)$ do not blow up at $x = 0$.

If $p(x)$ and $q(x)$ have Taylor series, then we can use the same idea as above to get series solution, but since we no longer expect a polynomial, the template for a series solution is a polynomial with infinite degree, whose template is

$$y(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$$

or in fancy sigma notation

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

but now we must find infinitely many numbers, namely $a_0, a_1, a_2, \ldots$. Of course this is impossible, but we can get the first few, and if in luck we can detect a pattern and find an expression for $a_n$ in terms of $n$. At any rate, we always get a recurrence relation that expresses $a_n$ in terms of $a_{n-1}, a_{n-2}$, so with a computer we can find thousands terms.

**Problem 24.1** For the diff.eq.

$$(1 - x)y''(x) + xy'(x) - y(x) = 0.$$

(a) Seek power series solution of the given differential equation at $x_0 = 0$, find the recurrence relation.

(b) Find the first four terms in each of two solutions $y_1(x), y_2(x)$ (unless the series terminates sooner)

**Solution of 24.1:**

**Step 1:** Write down the template

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and write down the series for $y'(x)$ and $y''(x)$

$$y'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$
Step 2: Plug into the diff.eq.

\[(1 - x)y''(x) + xy'(x) - y(x) =\]
\[(1 - x) \left( \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \right)\]
\[+ x \sum_{n=0}^{\infty} na_n x^{n-1}\]
\[- \sum_{n=0}^{\infty} a_n x^n .\]

Step 3: Open up all parentheses (and if the first and/or second terms are zero start the \(\sum\) later)

\[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\]
\[- \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}\]
\[+ \sum_{n=1}^{\infty} na_n x^n\]
\[- \sum_{n=0}^{\infty} a_n x^n .\]

Step 4. Rewrite each \(\sum\) so that we have \(x^n\) by shifting the summation (in the first \(\sum\) above we replace \(n\) by \(n+2\) to make the power \(x^n\) rather than \(x^{n-2}\); in the second \(\sum\) we replace \(n\) by \(n+1\).

Step 5: Open up all parentheses (and if the first and/or second terms are zero start the \(\sum\) later)

\[\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n\]
\[- \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n\]
\[+ \sum_{n=1}^{\infty} na_n x^n\]
\[- \sum_{n=0}^{\infty} a_n x^n .\]

Step 6 Collect into one \(\sum\) (and possibly a left-over at the beginning)
\[(2a_2 - a_0) + \sum_{n=1}^{\infty} \left( (n + 2)(n + 1)a_{n+2} - (n + 1)na_{n+1} + (n - 1)a_n \right) x^n \]

Set the initial term(s) to zero and the coefficient of \(x^n\), thereby getting the recurrence relation

\[a_2 = \frac{1}{2}a_0\]

\[(n + 2)(n + 1)a_{n+2} - (n + 1)na_{n+1} + (n - 1)a_n = 0.\]

This ends the first part.

**Step 7:**

To get the first 4 terms, we plug-in in turn \(n = 0, n = 1, n = 2\). We already know that \(a_2 = \frac{1}{2}a_0\).

When \(n = 1\) we get

\[(1 + 2)(1 + 1)a_{1+2} - (1 + 1)(1)a_{1+1} + (1 - 1)a_1 = 0.\]

So

\[6a_3 - 2a_2 = 0,\]

so \(a_3 = \frac{1}{3}a_2 = \frac{1}{3}a_0\).

When \(n = 2\) we get

\[(2 + 2)(2 + 1)a_{2+2} - (2 + 1)(2)a_{2+1} + (2 - 1)a_2 = 0.\]

So

\[12a_4 - 6a_3 + a_2 = 0.\]

So \(a_4 = \frac{1}{24}(6a_3 - a_2) = \frac{1}{24}(6\frac{1}{3} - \frac{1}{2})a_0 = \frac{1}{24}a_0\).

**Answer to 24.1** The recurrence relation for the coefficients \(a_n\) is

\[(n + 2)(n + 1)a_{n+2} - (n + 1)na_{n+1} + (n - 1)a_n = 0.\]

The first four coefficients are

\(a_1 = a_1, \ a_2 = \frac{1}{2}a_0, \ a_3 = \frac{1}{6}a_0, \ a_4 = \frac{1}{24}a_0.\)

**Remark** \(a_1\) never shows up later. So \(y_1(x) = x\) is a **one** (fundamental) solution! (very simple, by sheer luck) The other solution starts (taking \(a_0 = 1\)) \(y_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots\)