**Dr. Z.'s Calc4 Lecture 24 Handout: Series Solutions of Diff.Eqs.**

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The best kind of functions are **polynomials** that have the format

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \quad ,$$

for some finite $n$, called the **degree** of the polynomial. If we are really lucky, we may find a solution of a diff.eq. that is a polynomial. For example the diff.eq.

$$y''(x) + xy'(x) - 2y(x) = 0 \quad .$$

If someone told you that there is a solution that is a polynomial of degree 2, you can find it, by trying the **template**

$$y(x) = a_0 + a_1 x + a_2 x^2 \quad ,$$

featuring *undetermined coefficients* $a_0, a_1, a_2$. To find them, you first find expressions, in terms of $a_0, a_1, a_2$ for $y'(x)$ and $y''(x)$:

$$y'(x) = a_1 + 2a_2 x$$

$$y''(x) = 2a_2 \quad .$$

You now plug this template into the diff.eq. getting

$$2a_2 + x(a_1 + 2a_2 x) - 2(a_0 + a_1 x + a_2 x^2) = 2a_2 + a_1 x + 2a_2 x^2 - 2a_0 - 2a_1 x - 2a_2 x^2 \quad .$$

Now you collect coefficients getting

$$(2a_2 - 2a_0) - a_1 x \quad .$$

Since this is **identically zero**, each coefficient must be zero, so we have to solve the system

$$2a_2 - 2a_0 = 0 \quad , \quad a_1 = 0 \quad ,$$

whose solution is $a_0 = \mathit{anything}$, $a_1 = 0, a_2 = a_0$, so a solution is $a_0 + 0 \cdot x + a_0 x^2 = a_0(1 + x^2)$.

So we lucked out and found a polynomial solution, $f(x) = 1 + x^2$ and of course (since the diff.eq. is linear and homogeneous) any constant multiple of it.

This was luck! Consider a general **linear homogeneous second-order differential equation**

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) \; = \; 0 \quad .$$

For the sake of simplicity, let's look for solutions near $x = 0$ (we can always make a change of variable to move it to any desired point). If $P(x), Q(x), R(x)$ are nice (often these are polynomials) and $P(x)$ does not vanish at $x = 0$ (or, in general, at $x = x_0$) we can divide by $P(x)$ getting a simplified diff.eq. of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad ,$$

where $p(x)$ and $q(x)$ do not blow up at $x = 0$.

If $p(x)$ and $q(x)$ have Taylor series, then we can use the same idea as above to get **series solution**, but since we no longer expect a polynomial, the **template** for a series solution is a polynomial with **infinite** degree, whose template is

$$y(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \quad ,$$

or in fancy sigma notation

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad ,$$

but now we must find *infinitely* many numbers, namely $a_0, a_1, a_2, \ldots$. Of course this is impossible, but we can get the first few, and if in luck we can detect a pattern and find an expression for $a_n$ in terms of $n$. At any rate, we always get a **recurrence relation** that expresses $a_n$ in terms of $a_{n-1}, a_{n-2}$, so with a computer we can find thousands terms.

**Problem 24.1** For the diff.eq.

$$(1 - x)y''(x) + xy'(x) - y(x) = 0 \quad .$$

(a) Seek power series solution of the given differential equation at $x_0 = 0$, find the recurrence relation.

(b) Find the first four terms in each of two solutions $y_1(x), y_2(x)$ (unless the series terminates sooner)

**Solution of 24.1**:

**Step 1**: Write down the template

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and write down the series for $y'(x)$ and $y''(x)$

$$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

**Step 2**: Plug into the diff.eq.

$$
\begin{aligned}
&(1-x)y''(x) + xy'(x) - y(x) = \\
&(1-x)\left(\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}\right) \\
&\quad +x\sum_{n=0}^{\infty} n a_n x^{n-1} \\
&\quad -\sum_{n=0}^{\infty} a_n x^n \quad .
\end{aligned}
$$

**Step 3**: Open up all parentheses (and if the first and/or second terms are zero start the $\sum$ later)

$$
\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&-\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} \\
&+\sum_{n=1}^{\infty} n a_n x^n \\
&-\sum_{n=0}^{\infty} a_n x^n \quad .
\end{aligned}
$$

**Step 4.** Rewrite each $\sum$ so that we have $x^n$ by shifting the summation (in the first $\sum$ above we replace $n$ by $n+2$ to make the power $x^n$ rather than $x^{n-2}$, in the second $\sum$ we replace $n$ by $n+1$.

**Step 5**: Open up all parentheses (and if the first and/or second terms are zero start the $\sum$ later)

$$
\begin{aligned}
&\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n \\
&-\sum_{n=1}^{\infty} (n+1)n a_{n+1} x^n \\
&+\sum_{n=1}^{\infty} n a_n x^n \\
&-\sum_{n=0}^{\infty} a_n x^n \quad .
\end{aligned}
$$

**Step 6** Collect into one $\sum$ (and possibly a left-over at the beginning)

$$(2a_2 - a_0) + \sum_{n=1}^{\infty} (\, (n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n \,)\, x^n$$

Set the initial term(s) to zero and the coefficient of $x^n$, thereby getting the **recurrence relation**

$$a_2 = \frac{1}{2} a_0$$

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n = 0.$$

This ends the first part.

**Step 7**:

To get the first 4 terms, we plug-in in turn $n = 0, n = 1, n = 2$. We already know that $a_2 = \frac{1}{2}a_0$. When $n = 1$ we get

$$(1+2)(1+1)a_{1+2} - (1+1)(1)a_{1+1} + (1-1)a_1 = 0.$$

So

$$6a_3 - 2a_2 = 0 \quad ,$$

so $a_3 = \frac{1}{3}a_2 = \frac{1}{6}a_0$.

When $n = 2$ we get

$$(2+2)(2+1)a_{2+2} - (2+1)(2)a_{2+1} + (2-1)a_2 = 0.$$

So

$$12a_4 - 6a_3 + a_2 = 0.$$

So $a_4 = \frac{1}{12}(6a_3 - a_2) = \frac{1}{12}(6\frac{1}{6} - \frac{1}{2})a_0 = \frac{1}{24}a_0$.

**Answer to 24.1** The recurrence relation for the coefficients $a_n$ is

$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + (n-1)a_n = 0$.

The first four coefficients are

$a_1 = a_1$, $a_2 = \frac{1}{2}a_0$, $a_3 = \frac{1}{6}a_0$, $a_4 = \frac{1}{24}a_0$.

**Remark** $a_1$ never shows up later. So $y_1(x) = x$ is a **one** (fundamental) solution! (very simple, by sheer luck) The other solution starts (taking $a_0 = 1$) $y_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$.