Dr. Z.'s Calc4 Lecture 22 Handout: The Phase Plane: Linear Systems

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Important Facts

Suppose we have a **system** of two differential equations with two unknown functions, that is homog. and linear, so it can be written as

$$\mathbf{x}'(t) = \mathbf{P}\mathbf{x}(t) \quad ,$$

for some 2×2 matrix $\mathbf{P}$ of numbers. We have the following possible **scenarios**, regarding the **TYPE of Critical Point**, and **Stability** status (respectively)

Case Ia: Two real and distinct eigenvalues **BOTH** positive, $(r_1 > r_2 > 0)$:

Node ; Unstable

Case Ib: Two real and distinct eigenvalues **BOTH** negative, $(r_1 < r_2 < 0)$:

Node ; Asymptotically Stable

Case Ic: Two real and distinct eigenvalues of **OPPOSITE SIGN** $(r_2 < 0 < r_1)$:

Saddle Point ; Unstable

Case IIa: Repeated eigenvalue that is positive $(r_1 = r_2 > 0)$:

Proper or Improper Node ; Unstable

(Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

Case IIb: Repeated eigenvalue that is negative $(r_1 = r_2 < 0)$:

Proper or Improper Node ; Asymptotically Stable

(Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

Case IIIa: Complex eigenvectors ($r_1, r_2 = \lambda \pm i\mu$) with positive real part (i.e. $\lambda > 0$).

Spiral point ; Unstable

Case IIIb: Complex eigenvectors ($r_1, r_2 = \lambda \pm i\mu$) with negative real part, (i.e. $\lambda < 0$).

Spiral point ; Asymptotically Stable

Case IIIc: Complex eigenvectors ($r_1, r_2 = \lambda \pm i\mu$) with zero real part, (i.e. $\lambda = 0$).

Center ; Stable

Problem 22.1: Classify the critical point $(0, 0)$ as to type, and determine whether it is is stable, asymptotically stable, or unstable, for the following systems.

a,

$$\mathbf{x}'(t) = \begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix} \mathbf{x}(t)$$

b.

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & 2 \\ -6 & -5 \end{pmatrix} \mathbf{x}(t)$$

c.

$$\mathbf{x}'(t) = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix} \mathbf{x}(t)$$

d.

$$\mathbf{x}'(t) = \begin{pmatrix} -13 & 8 \\ -18 & -11 \end{pmatrix} \mathbf{x}(t)$$

e.

$$\mathbf{x}'(t) = \begin{pmatrix} 11 & 8 \\ -18 & -13 \end{pmatrix} \mathbf{x}(t)$$

f.

$$\mathbf{x}'(t) = \begin{pmatrix} 8 & 5 \\ -13 & -8 \end{pmatrix} \mathbf{x}(t)$$

g.

$$\mathbf{x}'(t) = \begin{pmatrix} -6 & -5 \\ 10 & 8 \end{pmatrix} \mathbf{x}(t)$$

h.

$$\mathbf{x}'(t) = \begin{pmatrix} 6 & 5 \\ -10 & -8 \end{pmatrix} \mathbf{x}(t)$$

Solution to Problem 22.1:

a.: The matrix of coefficients is

$$\begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}$$

the characteristic equation is

$$\det \begin{pmatrix} -2-r & -2 \\ 6 & 5-r \end{pmatrix} = (-2-r)(5-r)-(-2)(6) = (r+2)(r-5)+12 = r^2-3r-10+12 = r^2-3r+2 = 0 \quad .$$

To solve $r^2 - 3r + 2 = 0$ we factorize $(r-1)(r-2)$ getting two distinct eigenvalues $r_1 = 2, r_2 = 1$, both real and positive. This is case Ia, so the critical point $(0,0)$ is a **node**, and it is **unstable**.

Ans to a.: The critical point $(0,0)$ is a **node**, and it is **unstable**.

b.: The matrix of coefficients is

$$\begin{pmatrix} 2 & 2 \\ -6 & -5 \end{pmatrix}$$

the characteristic equation is

$$\det \begin{pmatrix} 2-r & 2 \\ -6 & -5-r \end{pmatrix} = (2-r)(-5-r)-(2)(-6) = (r-2)(r+5)+12 = r^2+3r-10+12 = r^2+3r+2 = 0 \quad .$$

To solve $r^2 + 3r + 2 = 0$ we factorize $(r+1)(r+2)$ getting two distinct eigenvalues $r_1 = -2, r_2 = -1$, both real and negative. This is case Ib, so the critical point $(0,0)$ is a **node**, and it is **asymptotically stable**.

Ans. to b.: The critical point $(0,0)$ is a **node**, and it is **asymptotically stable**.

c.: The matrix of coefficients is

$$\begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix}$$

the characteristic equation is

$$\det \begin{pmatrix} -10-r & -6 \\ 18 & 11-r \end{pmatrix} = (-10-r)(11-r)-(-6)(18) = (r+10)(r-11)+108 = r^2-r-110+108 = r^2-r-2 = 0 \quad .$$

To solve $r^2 - r - 2 = 0$ we factorize $(r-2)(r+1)$ getting two distinct eigenvalues $r_1 = -1, r_2 = 2$, that are of **opposite sign**.

This is case Ic, so the critical point $(0,0)$ is a **saddle point**, and it is **unstable**.

Ans. to c.: The critical point $(0,0)$ is a **saddle point**, and it is **unstable**.

d.: The matrix of coefficients is

$$\begin{pmatrix} 13 & 8 \\ -18 & -11 \end{pmatrix}$$

the characteristic equation is

$$\det \begin{pmatrix} 13-r & 8 \\ -18 & -11-r \end{pmatrix} = (13-r)(-11-r)-(8)(-18) = (r-13)(r+11)+144 = r^2-2r-143+144 = r^2-2r+1 = 0 \quad .$$

To solve $r^2 - 2r + 1 = 0$ we factorize $(r-1)^2 = 0$ getting a double eigenvalues $r_1 = 1$, that is **positive**.

This is case IIa, so the critical point $(0,0)$ is a **node**, and it is **unstable**.

(Note: it turns out that the eigenspace is one-dimensional (you do it!), so it it an improper node)

Ans. to d.: The critical point $(0,0)$ is an **improper node**, and it is **unstable**.

e.: The matrix of coefficients is

$$\begin{pmatrix} 11 & 8 \\ -18 & -13 \end{pmatrix}$$

the characteristic equation is

$$\det\begin{pmatrix} 11-r & 8 \\ -18 & -13-r \end{pmatrix} = (11-r)(-13-r)-(8)(-18) = (r-11)(r+13)+144 = r^2+2r-143+144 = r^2+2r+1 = 0 \quad .$$

To solve $r^2+2r+1=0$ we factorize $(r+1)^2=0$ getting a double eigenvalues $r_1=-1$, that is **negative**.

This is case IIb, so the critical point $(0,0)$ is a **node**, and it is **asymptotically stable**.

(Note: it turns out that the eigenspace is one-dimensional (you do it!), so it it an improper node)

Ans. to e.: The critical point $(0,0)$ is an **improper node**, and it is **asymptotically stable**.

f.: The matrix of coefficients is

$$\begin{pmatrix} 8 & 5 \\ -13 & -8 \end{pmatrix}$$

the characteristic equation is

$$\det\begin{pmatrix} 8-r & 5 \\ -13 & -8-r \end{pmatrix} = (8-r)(-8-r)-(5)(-13) = (r-8)(r+8)+65 = r^2-64+65 = r^2+1 = 0 \quad .$$

The roots of $r^2+1=0$ are $0 \pm i$, so we have **complex roots** with $\lambda = 0$.

This is case IIIc, so the critical point $(0,0)$ is a **center**, and it is **stable**.

Ans. to f.: The critical point $(0,0)$ is **center**, and it is **stable**.

g.: The matrix of coefficients is

$$\begin{pmatrix} -6 & -5 \\ 10 & 8 \end{pmatrix}$$

the characteristic equation is

$$\det\begin{pmatrix} -6-r & 5 \\ 10 & 8-r \end{pmatrix} = (-6-r)(8-r)-(-5)(10) = (r+6)(r-8)+50 = r^2-2r-48+50 = r^2-2r+2 = 0 \quad .$$

The roots of $r^2-2r+2=0$ are $\frac{2\pm\sqrt{(-2)^2-4(1)(2)}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$

So we have complex roots with $\lambda = 1$, positive.

This is case IIIa, so the critical point $(0,0)$ is a **spiral point**, and it is **unstable**.

Ans. to g.: The critical point $(0,0)$ is a **spiral point** , and it is **unstable**.

h.: The matrix of coefficients is

$$\begin{pmatrix} 6 & 5 \\ -10 & -8 \end{pmatrix}$$

the characteristic equation is

$$\det \begin{pmatrix} 6-r & 5 \\ -10 & -8-r \end{pmatrix} = (6-r)(-8-r)-(5)(-10) = (r-6)(r+8)+50 = r^2+2r-48+50 = r^2+2r+2 = 0 \quad .$$

The roots of $r^2+2r+2=0$ are $\frac{-2\pm\sqrt{(2)^2-4(1)(2)}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$

So we have complex roots with $\lambda = -1$, negative.

This is case IIIb, so the critical point $(0,0)$ is a **spiral point**, and it is **asymptotically stable**.

Ans. to h.: The critical point $(0,0)$ is a **spiral point** , and it is **asymptotically stable**.