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**Important Facts**

Suppose we have a system of two differential equations with two unknown functions, that is homog. and linear, so it can be written as

\[ x'(t) = Px(t) \]

for some $2 \times 2$ matrix $P$ of numbers. We have the following possible scenarios, regarding the **TYPE of Critical Point**, and **Stability** status (respectively)

**Case Ia**: Two real and distinct eigenvalues BOTH positive, $(r_1 > r_2 > 0)$:

Node ; Unstable

**Case Ib**: Two real and distinct eigenvalues BOTH negative, $(r_1 < r_2 < 0)$:

Node ; Asymptotically Stable

**Case Ic**: Two real and distinct eigenvalues of OPPOSITE SIGN $(r_2 < 0 < r_1)$:

Saddle Point ; Unstable

**Case IIa**: Repeated eigenvalue that is positive $(r_1 = r_2 > 0)$:

Proper or Improper Node ; Unstable

(Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

**Case IIb**: Repeated eigenvalue that is negative $(r_1 = r_2 < 0)$:

Proper or Improper Node ; Asymptotically Stable

(Note: if the eigenspace is two-dimensional, it is a proper node, if it is one-dimensional, it is improper)

**Case IIIa**: Complex eigenvectors $(r_1, r_2 = \lambda \pm i\mu)$ with positive real part (i.e. $\lambda > 0$).

Spiral point ; Unstable

**Case IIIb**: Complex eigenvectors $(r_1, r_2 = \lambda \pm i\mu)$ with negative real part, (i.e. $\lambda < 0$).

Spiral point ; Asymptotically Stable
Case IIIc: Complex eigenvectors \( r_1, r_2 = \lambda \pm i\mu \) with zero real part, (i.e. \( \lambda = 0 \)).

Center; Stable

Problem 22.1: Classify the critical point \((0,0)\) as to type, and determine whether it is stable, asymptotically stable, or unstable, for the following systems.

\( a. \)
\[
x'(t) = \begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix} x(t)
\]

\( b. \)
\[
x'(t) = \begin{pmatrix} 2 & 2 \\ -6 & -5 \end{pmatrix} x(t)
\]

\( c. \)
\[
x'(t) = \begin{pmatrix} -10 & -6 \\ 18 & 11 \end{pmatrix} x(t)
\]

\( d. \)
\[
x'(t) = \begin{pmatrix} -13 & 8 \\ -18 & -11 \end{pmatrix} x(t)
\]

\( e. \)
\[
x'(t) = \begin{pmatrix} 11 & 8 \\ -18 & -13 \end{pmatrix} x(t)
\]

\( f. \)
\[
x'(t) = \begin{pmatrix} 8 & 5 \\ -13 & -8 \end{pmatrix} x(t)
\]

\( g. \)
\[
x'(t) = \begin{pmatrix} -6 & -5 \\ 10 & 8 \end{pmatrix} x(t)
\]

\( h. \)
\[
x'(t) = \begin{pmatrix} 6 & 5 \\ -10 & -8 \end{pmatrix} x(t)
\]

Solution to Problem 22.1:

\( a. \): The matrix of coefficients is
\[
\begin{pmatrix} -2 & -2 \\ 6 & 5 \end{pmatrix}
\]
the characteristic equation is
\[
\det \begin{pmatrix} -2 - r & -2 \\ 6 & 5 - r \end{pmatrix} = (-2-r)(5-r)-(-2)(6) = (r+2)(r-5)+12 = r^2-3r-10+12 = r^2-3r+2 = 0
\]
To solve $r^2 - 3r + 2 = 0$ we factorize $(r - 1)(r - 2)$ getting two distinct eigenvalues $r_1 = 2, r_2 = 1$, both real and positive. This is case Ia, so the critical point $(0, 0)$ is a node, and it is unstable.

**Ans to a.** The critical point $(0, 0)$ is a node, and it is unstable.

**b.** The matrix of coefficients is

$$
\begin{pmatrix}
2 & 2 \\
-6 & -5
\end{pmatrix}
$$

the characteristic equation is

$$\det\begin{pmatrix}
2 - r & 2 \\
-6 & -5 - r
\end{pmatrix} = (2-r)(-5-r)-(2)(-6) = (r-2)(r+5)+12 = r^2+3r-10+12 = r^2+3r+2 = 0 .
$$

To solve $r^2 + 3r + 2 = 0$ we factorize $(r + 1)(r + 2)$ getting two distinct eigenvalues $r_1 = -2, r_2 = -1$, both real and negative. This is case Ib, so the critical point $(0, 0)$ is a node, and it is asymptotically stable.

**Ans. to b.** The critical point $(0, 0)$ is a node, and it is asymptotically stable.

**c.** The matrix of coefficients is

$$
\begin{pmatrix}
-10 & -6 \\
18 & 11
\end{pmatrix}
$$

the characteristic equation is

$$\det\begin{pmatrix}
-10 - r & -6 \\
18 & 11 - r
\end{pmatrix} = (-10-r)(11-r)-(-6)(18) = (r+10)(r-11)+108 = r^2-r-110+108 = r^2-r-2 = 0 .
$$

To solve $r^2 - r - 2 = 0$ we factorize $(r - 2)(r + 1)$ getting two distinct eigenvalues $r_1 = -1, r_2 = 2$, that are of opposite sign.

This is case Ic, so the critical point $(0, 0)$ is a saddle point, and it is unstable.

**Ans. to c.** The critical point $(0, 0)$ is a saddle point, and it is unstable.

**d.** The matrix of coefficients is

$$
\begin{pmatrix}
13 & 8 \\
-18 & -11
\end{pmatrix}
$$

the characteristic equation is

$$\det\begin{pmatrix}
13 - r & 8 \\
-18 & -11 - r
\end{pmatrix} = (13-r)(-11-r)-(8)(-18) = (r-13)(r+11)+144 = r^2-2r-143+144 = r^2-2r+1 = 0 .
$$

To solve $r^2 - 2r + 1 = 0$ we factorize $(r - 1)^2 = 0$ getting a double eigenvalues $r_1 = 1$, that is positive.

This is case IIa, so the critical point $(0, 0)$ is a node, and it is unstable.

(Note: it turns out that the eigenspace is one-dimensional (you do it!), so it it an improper node)
Ans. to d.: The critical point \((0,0)\) is an improper node, and it is unstable.

e.: The matrix of coefficients is
\[
\begin{pmatrix}
11 & 8 \\
-18 & -13
\end{pmatrix}
\]
the characteristic equation is
\[
\det \begin{pmatrix}
11 - r & 8 \\
-18 & -13 - r
\end{pmatrix} = (11-r)(-13-r)-8(-18) = (r-11)(r+13)+144 = r^2+2r-143+144 = r^2+2r+1 = 0
\]
To solve \(r^2 + 2r + 1 = 0\) we factorize \((r + 1)^2 = 0\) getting a double eigenvalues \(r_1 = -1\), that is negative.

This is case IIb, so the critical point \((0,0)\) is a node, and it is asymptotically stable.

(Note: it turns out that the eigenspace is one-dimensional (you do it!), so it it an improper node)

Ans. to e.: The critical point \((0,0)\) is an improper node, and it is asymptotically stable.

f.: The matrix of coefficients is
\[
\begin{pmatrix}
8 & 5 \\
-13 & -8
\end{pmatrix}
\]
the characteristic equation is
\[
\det \begin{pmatrix}
8 - r & 5 \\
-13 & -8 - r
\end{pmatrix} = (8-r)(-8-r)-(5)(-13) = (r-8)(r+8)+65 = r^2-64+65 = r^2+1 = 0
\]
The roots of \(r^2 + 1 = 0\) are \(0 \pm i\), so we have complex roots with \(\lambda = 0\).

This is case IIIc, so the critical point \((0,0)\) is a center, and it is stable.

Ans. to f.: The critical point \((0,0)\) is center, and it is stable.

g.: The matrix of coefficients is
\[
\begin{pmatrix}
-6 & -5 \\
10 & 8
\end{pmatrix}
\]
the characteristic equation is
\[
\det \begin{pmatrix}
-6 - r & -5 \\
10 & 8 - r
\end{pmatrix} = (-6-r)(8-r)-(-5)(10) = (r+6)(r-8)+50 = r^2-2r-48+50 = r^2-2r+2 = 0
\]
The roots of \(r^2 - 2r + 2 = 0\) are \(\frac{2 \pm \sqrt{(-2)^2-4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i\)

So we have complex roots with \(\lambda = 1\), positive.

This is case IIIa, so the critical point \((0,0)\) is a spiral point, and it is unstable.

Ans. to g.: The critical point \((0,0)\) is a spiral point, and it is unstable.
h.: The matrix of coefficients is
\[
\begin{pmatrix}
6 & 5 \\
-10 & -8
\end{pmatrix}
\]
the characteristic equation is
\[
\det\begin{pmatrix}
6-r & 5 \\
-10 & -8-r
\end{pmatrix} = (6-r)(-8-r)-(5)(-10) = (r-6)(r+8)+50 = r^2+2r-48+50 = r^2+2r+2 = 0 .
\]
The roots of \(r^2 + 2r + 2 = 0\) are
\[
\frac{-2 \pm \sqrt{(2)^2-4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i
\]
So we have complex roots with \(\lambda = -1\), negative.

This is case IIIb, so the critical point \((0, 0)\) is a spiral point, and it is asymptotically stable.

Ans. to h.: The critical point \((0, 0)\) is a spiral point, and it is asymptotically stable.