The Case of Complex Roots When Solving Homogeneous Linear Systems with Constant Coefficients

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It often happens that when we try to find solutions of homog. systems of linear diff.eqs. with constant coefficients, of the form

\[ x(t) = v e^{rt}, \]

where \( v \) is a CONSTANT vector and \( r \) is some number, and follow the procedure of Lecture 19, trying to find the eigenvalues of the matrix \( P \), the characteristic equation has complex roots. Since the matrix \( P \) has real entries, the roots come in complex-conjuate pairs \( \lambda \pm i\mu \). The good news is that we only need to consider one eigenvalue of each of these pairs (so for systems of 2 equations and 2 unknown functions, just one). The bad news is that we need to use complex calculations, always keeping in mind that \( i^2 = -1 \).

**Problem 20.1**

Find the general solution of the system

\[ x'(t) = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x(t). \]

**Step 1.** Write down the matrix of coefficients, and set-up the characteristic equation.

\[ P = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \]

\[ \det \left( \begin{array}{cc} 1 - r & -5 \\ 1 & -3 - r \end{array} \right) = 0. \]

**Step 2.** Compute the determinant, and solve the characteristic equation, finding the eigenvalues.

\[ (1 - r)(-3 - r) - (-5)(1) = 0, \]

\[ (r - 1)(r + 3) + 5 = 0, \]

\[ r^2 + 2r - 3 + 5 = 0, \]

\[ r^2 + 2r + 2 = 0. \]

Using the famous formula for finding the roots of a quadratic equation

\[ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \]

we get

\[ r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i. \]
Step 3. We only need to find the eigenvector corresponding to \( r_1 = -1 + i \). When \( r = -1 + i \),
\[
\begin{pmatrix}
1 - r & -5 \\
1 & -3 - r
\end{pmatrix}
\]
becomes
\[
\begin{pmatrix}
1 - (-1 + i) & -5 \\
1 & -3 - (-1 + i)
\end{pmatrix} = \begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix}.
\]
We have to find a vector \((a_1, a_2)^T\) such that
\[
\begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Spelling it out:
\[
(2 - i)a_1 - 5a_2 = 0, \quad a_1 - (2 + i)a_2 = 0.
\]
These two equations are multiple of each other, so it is enough to consider one of them, so let’s pick the second, that is simpler. We have \(a_1 = (2 + i)a_2\). So taking \(a_2 = 1\) we get that \(a_1 = 2 + i\).

So an eigenvector corresponding to \( r_1 = -1 + i \) is \(v_1 = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix}\).

Step 4. The (one, specific) solution that we found so far is
\[
x(t) = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix} e^{(-1+i)t}
\]
Now we write
\[
e^{(-1+i)t} = e^{-t}e^{it},
\]
and use Euler’s famous formula
\[
e^{it} = \cos t + i \sin t,
\]
to get
\[
x(t) = e^{-t} \begin{pmatrix} 2 + i \\ 1 \end{pmatrix} (\cos t + i \sin t)
\]
This equals
\[
x(t) = e^{-t} \begin{pmatrix} (2 + i)(\cos t + i \sin t) \\ (\cos t + i \sin t) \end{pmatrix}
\]
Doing the complex algebra, this is
\[
x(t) = e^{-t} \begin{pmatrix} 2 \cos t + 2i(\sin t) + i \cos t - \sin t \\ \cos t + i \sin t \end{pmatrix} = e^{-t} \begin{pmatrix} (2 \cos t - \sin t) + i(2 \sin t + \cos t) \\ \cos t + i \sin t \end{pmatrix}
\]
We now separate the real and imaginary parts
\[
x(t) = e^{-t} \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + ie^{-t} \begin{pmatrix} 2 \sin t + \cos t \\ \sin t \end{pmatrix}.\]
Obviously the real and imaginary parts are \textbf{linearly independent}, so we found two independent solutions

\[
x_1(t) = e^{-t} \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix}, \\
x_2(t) = e^{-t} \begin{pmatrix} 2 \sin t + \cos t \\ \sin t \end{pmatrix}.
\]

The \textbf{general solutions} is simply \( c_1 x_1(t) + c_2 x_2(t), \) where \( c_1, c_2 \) are \textbf{arbitrary constants}.

In this problem it is

\[
x(t) = c_1 e^{-t} \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \sin t + \cos t \\ \sin t \end{pmatrix}.
\]

**Ans. to 20.1:** \( x(t) = c_1 e^{-t} \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \sin t + \cos t \\ \sin t \end{pmatrix} \)

In scalar notation:

\[
x_1(t) = e^{-t}(c_1(2 \cos t - \sin t) + c_2(\cos t + 2 \sin t)) , \quad x_2(t) = e^{-t}(c_1 \cos t + c_2 \sin t) .
\]

**Problem 20.2**

Solve the initial value system

\[
x'(t) = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x(t) , \quad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

**Steps 1-4.** Find the general solution exactly as in Problem 20.1.

\[
x(t) = c_1 e^{-t} \begin{pmatrix} 2 \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \sin t + \cos t \\ \sin t \end{pmatrix}.
\]

**Step 5:** Plug in \( t = 0 \) (or, in general \( t = t_0 \), if the initial condition is no at 0).

\[
x(0) = c_1 e^{-0} \begin{pmatrix} 2 \cos 0 - \sin 0 \\ \cos 0 \end{pmatrix} + c_2 e^{-0} \begin{pmatrix} 2 \sin 0 + \cos 0 \\ \sin 0 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix}.
\]

**Step 6:** Set it equal to the vector \( x(0) \) given by the problem,

\[
\begin{pmatrix} 2c_1 + c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

and spelled-out:

\[
2c_1 + c_2 = 1 , \quad c_1 = 1 ,
\]

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and solve for $c_1, c_2$.

Here $c_1 = 1$ and $c_2 = -1$.

**Step 7.** Go back to the general solution and enter the $c_1, c_2$ that you just found.

$$x(t) = e^{-t} \left( \begin{array}{c} 2 \cos t - \sin t \\ \cos t \end{array} \right) - e^{-t} \left( \begin{array}{c} 2 \sin t + \cos t \\ \sin t \end{array} \right).$$

$$= e^{-t} \left( \begin{array}{c} \cos t - 3 \sin t \\ \cos t - \sin t \end{array} \right)$$

**Ans. to Problem 20.2:** $x(t) = e^{-t} \left( \begin{array}{c} \cos t - 3 \sin t \\ \cos t - \sin t \end{array} \right)$ or

$$x_1(t) = e^{-t}(\cos t - 3 \sin t) \quad , \quad x_2(t) = e^{-t}(\cos t - \sin t).$$