Dr. Z.’s Calc4 Lecture 17 Handout: Introducing Systems of First Order Linear Equations; Review of Matrices

By Doron Zeilberger

So far we considered one diff.eq. with one unknown function to be found, usually written $y(t)$ or $y(x)$, where $t$ or $x$ were the independent variable and $y$ was the dependent variable.

Often, in applications, we have several (say $n$) differential equations with several (usually the same number, $n$) of unknown functions, called $x_1(t), x_2(t), \ldots, x_n(t)$. We only consider first order equations, i.e. when we do systems, we only have the first derivative show up.

The general format of a System of First-Order Differential Equations with $n$ functions to look for, $x_1(t), \ldots, x_n(t)$ is

\[
x_1'(t) = F_1(t, x_1(t), x_2(t), \ldots, x_n(t)) ,
\]
\[
x_2'(t) = F_2(t, x_1(t), x_2(t), \ldots, x_n(t)) ,
\]
\[\ldots
\]
\[
x_n'(t) = F_n(t, x_1(t), x_2(t), \ldots, x_n(t)) .
\]

Here $F_1(t, x_1, \ldots, x_n), \ldots, F_n(t, x_1, \ldots, x_n)$ are some (possibly very complicated) multivariable functions of $n + 1$ arguments.

If we specify initial conditions

\[
x_1(t_0) = x_1^0 , \quad x_2(t_0) = x_2^0 , \quad \ldots , \quad x_n(t_0) = x_n^0 ,
\]

then we have an initial value problem.

Of course, it is usually not possible to get an exact solution, in terms of a formula, and the best that we can hope for is to find good approximations, on the computer, but, in an abstract sense, we know that solutions exist, if the functions $F_1, F_2, \ldots, F_n$ featured in the system, are not too crazy.

We have


If the functions $F_1, \ldots, F_n$ and all their partial derivatives are continuous (do not blow up and have no breaks) in a box-like region $R$ of the $(n + 1)$ dimensional $tx_1 \ldots x_n$ space containing the point $(t_0, x_1^0, \ldots, x_n^0)$. Then there is an interval $|t - t_0| < h$ in which there is unique solution of the above initial value problem.
An important special case of systems of Diff.Eqs. are **Linear Systems of Diff.Eq.s**. whose format is

\[ x'_1(t) = p_{11}(t)x_1(t) + \ldots + p_{1n}(t)x_n + g_1(t) \]
\[ x'_2(t) = p_{21}(t)x_1(t) + \ldots + p_{2n}(t)x_n + g_2(t) \]
\[ \ldots \]
\[ \ldots \]
\[ x'_n(t) = p_{n1}(t)x_1(t) + \ldots + p_{nn}(t)x_n + g_n(t) \]

If all the \( g_i(t) \) are 0 then we have a **homogeneous system**.

If all the coefficient functions \( p_{ij}(t) \) are continuous in an interval \( I \), then we are guaranteed a solution satisfying any initial conditions.

**Converting ONE Higher-Order Diff.Eq. to a FIRST-ORDER System**

Whenever we have one diff.eq. of the format

\[ y^{(n)}(t) = F(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \]

there is a quick way to make it into a first order system, as follows. Assuming that we already know \( y(t) \), we define

\[ x_1(t) = y(t) \]
\[ x_2(t) = y'(t) \]
\[ x_3(t) = y''(t) \]
\[ \ldots \]
\[ x_n(t) = y^{(n-1)}(t) \]

then

\[ x'_1(t) = x_2(t) \]
\[ x'_2(t) = x_3(t) \]
\[ x'_3(t) = x_4(t) \]
\[ \ldots \]
\[ x'_{n-1}(t) = x_n(t) \]
\[ x'_n(t) = F(t, x_1, \ldots, x_n) \]

Note that only the last equation is “complicated”, the first \( n - 1 \) ones are very simple.

**Problem 17.1**: Convert the following third-order diff.eq. to a system of first-order diff.eqs

\[ y'''(t) = \cos(y''(t) + t^3 + y'(t)y(t)) + \sin(y(t)) \]

**Solutions to 17.1**: The first \( n - 1 \) equations are **always** the same. Here \( n = 3 \) (since it is a third-order diff.eq.) so our first two equations are

\[ x'_1(t) = x_2(t) \]
\[ x'_2(t) = x_3(t) \]

and to get the last one, you replace \( y'''(t) \) by \( x'_3(t) \) and \( y''(t) \) by \( x_3(t) \), \( y'(t) \) by \( x_2(t) \), and \( y(t) \) by \( x_1(t) \). In this problem

\[ x'_3(t) = \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin(x_1(t)) \]

**Ans. to 17.1:**

\[ x'_1(t) = x_2(t) \quad x'_2(t) = x_3(t) \quad x'_3(t) = \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin(x_1(t)) \]

But sometimes one can go the other way. Given a first-order system, we can solve it using what we know about higher-order diff.eq. Lucky for us, we only need to do it for linear systems with constant coefficients.

**Problem 17.2** Solve the initial value problem for the system

\[ x'_1(t) = -2x_1(t) + x_2(t) \quad x'_2(t) = x_1(t) - 2x_2(t) \quad x_1(0) = 2 \quad x_2(0) = 3 \]

**Solution to 17.2**

**Step 1:** Use the first equation, and algebra, to express \( x_2(t) \) in terms of \( x_1(t) \) (and its derivative \( x'_1(t) \)).

\[ x_2(t) = x'_1(t) + 2x_1(t) \]

**Step 2:** Substitute this into second equation:

\[ (x'_1(t) + 2x_1(t))' = x_1(t) - 2(x'_1(t) + 2x_1(t)) \]

**Step 3:** Use calculus and algebra to simplify

\[ x''_1(t) + 2x'_1(t) = x_1(t) - 2x'_1(t) - 4x_1(t) \]

\[ x''_1(t) + 4x'_1(t) + 3x_1(t) = 0 \]

**Step 4:** Go back to Step 1 and plug \( t = 0 \) and use algebra to find \( x'_1(0) \):

\[ x_2(0) = x'_1(0) + 2x_1(0) \]

\[ x'_1(0) = x_2(0) - 2x_1(0) = 3 - 2 \cdot 2 = 3 - 4 = -1 \]

**Step 5:** Solve the initial value problem

\[ x''_1(t) + 4x'_1(t) + 3x_1(t) = 0 \quad x_1(0) = 2 \quad x'_1(0) = -1 \]
We get \( x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \)

**Step 6:** Go back to Step 1 and find out what is \( x_2(t) \):

\[
x_2(t) = x_1'(t) + 2x_1(t) = \left( \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \right)' + 2\left( \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \right) = -\frac{5}{2}e^{-t} + \frac{3}{2}e^{-3t} + 5e^{-t} - e^{-3t} = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}
\]

**Ans. to 17.2:** \( x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t} \), \( x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t} \).

**Review of Vectors and Matrices**

Look it up in wikipedia. In Maple you use the package LinearAlgebra. Look up the commands Matrix, Inverse, Multiply.