Dr. Z.'s Calc4 Lecture 17 Handout: Introducing Systems of First Order Linear Equations; Review of Matrices

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So far we considered **one** diff.eq. with **one** unknown function to be found, usually written $y(t)$ or $y(x)$, where t or x were the *independent variable* and y was the *dependent variable.*

Often, in applications, we have **several** (say n) differential equations with **several** (usually the same number, n) of unknown functions, called $x_1(t), x_2(t), \quad , x_n(t)$. We only consider **first order** equations, i.e. when we do systems, we only have the **first** derivative show up.

The **general format** of a **System of First-Order Differential Equations** with n functions to look for, $x_1(t), \ldots, x_n(t)$ is

$$x_1'(t) = F_1(t, x_1(t), x_2(t), \ldots, x_n(t)) \quad ,$$

$$x_2'(t) = F_2(t, x_1(t), x_2(t), \ldots, x_n(t)) \quad ,$$

$$\ldots$$

$$\ldots$$

$$x_n'(t) = F_n(t, x_1(t), x_2(t), \ldots, x_n(t)) \quad .$$

Here $F_1(t, x_1, \ldots, x_n), \ldots, F_n(t, x_1, \ldots, x_n)$ are some (possibly very complicated) multivariable functions of $n+1$ arguments.

If we specify initial conditions

$$x_1(t_0) = x_1^0 \quad , \quad x_2(t_0) = x_2^0 \quad , \quad \ldots \quad , \quad x_n(t_0) = x_n^0 \quad ,$$

then we have an *initial value problem.*

Of course, it is usually not possible to get an exact solution, in terms of a formula, and the best that we can hope for is to find good approximations, on the computer, **but**, in an abstract sense, we know that solutions exist, if the functions $F_1, F_2, \ldots, F_n$ featured in the system, are not too crazy.

We have

Existence Theorem for Systems of First Order Diff. Eqs.

If the functions $F_1, \ldots, F_n$ and all their partial derivatives are continuous (do not blow up and have no breaks) in a box-like region R of the $(n+1)$ dimensional $tx_1 \ldots x_n$ space containing the point $(t_0, x_1^0, \ldots, x_n^0)$. Then there is an interval $|t - t_0| < h$ in which there is unique solution of the above initial value problem.

An important special case of systems of Diff.Eqs. are **Linear** Systems of Diff.Eq.s. whose format is

$$x_1'(t) = p_{11}(t)x_1(t) + \ldots + p_{1n}(t)x_n + g_1(t)$$
$$x_2'(t) = p_{21}(t)x_1(t) + \ldots + p_{2n}(t)x_n + g_2(t)$$
$$\ldots$$
$$\ldots$$
$$x_n'(t) = p_{n1}(t)x_1(t) + \ldots + p_{nn}(t)x_n + g_n(t) \quad .$$

If all the $g_i(t)$ are 0 then we have a **homogeneous system**.

If all the coefficient functions $p_{ij}(t)$ are continuous in an interval I, then we are guaranteed a solution satisfying any initial conditions.

Converting ONE Higher-Order Diff.Eq. to a FIRST-ORDER System

Whenever we have one diff.eq. of the format

$$y^{(n)}(t) = F(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \quad ,$$

there is a quick way to make it into a first order system, as follows. Assuming that we already know $y(t)$, we define

$$x_1(t) = y(t) \quad , \quad x_2(t) = y'(t) \quad , \quad x_3(t) = y''(t) \quad , \quad \ldots \quad , \quad x_n(t) = y^{(n-1)}(t),$$

then

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = x_3(t)$$
$$x_3'(t) = x_4(t)$$
$$\ldots$$
$$x_{n-1}'(t) = x_n(t)$$
$$x_n'(t) = F(t, x_1, \ldots, x_n) \quad .$$

Note that only the last equation is "complicated", the first $n-1$ ones are very simple.

Problem 17.1: Convert the following third-order diff.eq. to a system of first-order diff.eqs

$$y'''(t) = \cos(y''(t) + t^3 + y'(t)y(t)) + \sin(y(t))$$

Solutions to 17.1: The first $n-1$ equations are **always** the same. Here $n = 3$ (since it is a third-order diff.eq.) so our first two equations are

$$x_1'(t) = x_2(t)$$

$$x_2'(t) = x_3(t)$$

and to get the last one, you replace $y'''(t)$ by $x_3'(t)$ and $y''(t)$ by $x_3(t)$, $y'(t)$ by $x_2(t)$, and $y(t)$ by $x_1(t)$. In this problem

$$x_3'(t) = \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin(x_1(t)) \quad .$$

Ans. to 17.1:

$$x_1'(t) = x_2(t) \quad , \quad x_2'(t) = x_3(t) \quad , \quad x_3'(t) = \cos(x_3(t) + t^3 + x_2(t)x_1(t)) + \sin x_1(t) \quad .$$

But sometimes one can go the other way. Given a first-order system, we can solve it using what we know about higher-order diff.eq. Lucky for us, we only need to do it for linear systems with constant coefficients.

Problem 17.2 Solve the initial value problem for the system

$$x_1'(t) = -2x_1(t) + x_2(t) \quad , \quad x_2'(t) = x_1(t) - 2x_2(t) \quad ; \quad x_1(0) = 2 \quad , \quad x_2(0) = 3 \quad .$$

Solution to 17.2

Step 1: Use the first equation, and algebra, to express $x_2(t)$ in terms of $x_1(t)$ (and its derivative $x_1'(t)$).

$$x_2(t) = x_1'(t) + 2x_1(t)$$

Step 2: Substitute this into second equation:

$$(x_1'(t) + 2x_1(t))' = x_1(t) - 2(x_1'(t) + 2x_1(t))$$

Step 3: Use calculus and algbera to simplify

$$x_1''(t) + 2x_1'(t) = x_1(t) - 2x_1'(t) - 4x_1(t)$$

$$x_1''(t) + 4x_1'(t) + 3x_1(t) = 0$$

Step 4: Go back to Step 1 and plug-in $t = 0$ and use algebra to find $x_1'(0)$:

$$x_2(0) = x_1'(0) + 2x_1(0) \quad .$$

$$x_1'(0) = x_2(0) - 2x_1(0) = 3 - 2 \cdot 2 = 3 - 4 = -1 \quad .$$

Step 5: Solve the initial value problem

$$x_1''(t) + 4x_1'(t) + 3x_1(t) = 0 \quad , \quad x_1(0) = 2 \quad , \quad x_1'(0) = -1$$

We get $x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$

Step 6: Go back to Step 1 and find out what is $x_2(t)$:

$$x_2(t) = x_1'(t) + 2x_1(t) = (\frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t})' + 2(\frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}) = -\frac{5}{2}e^{-t} + \frac{3}{2}e^{-3t} + 5e^{-t} - e^{-3t} = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

Ans. to 17.2: $x_1(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$, $x_2(t) = \frac{5}{2}e^{-t} + \frac{1}{2}e^{-3t}$.

Review of Vectors and Matrices

Look it up in wikipedia. In Maple you use the package LinearAlgebra . Look up the commands Matrix, Inverse, Multiply.