Important Theorem (Complicated Version)

If the functions $p(t), q(t), g(t)$ are continuous on an open interval $I$, and if $y_1(t)$ and $y_2(t)$ are independent solutions of the **homogeneous** diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

then a particular solution of the **inhomogeneous** diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t),$$

is given by

$$-y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)} ds,$$

where $W(y_1, y_2)(t) = y_1(t)y'_2(t) - y_2(t)y'_1(t)$.

Important Theorem (Simple Version)

If the functions $p(t), q(t), g(t)$ are continuous on an open interval $I$, and if $y_1(t)$ and $y_2(t)$ are independent solutions of the **homogeneous** diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0,$$

then a particular solution of the **inhomogeneous** diff.eq.

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t),$$

is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where $u_1(t), u_2(t)$ are two functions whose derivatives satisfy the system of two equations

$$u'_1(t)y_1(t) + u'_2(t)y_2(t) = 0,$$

$$u'_1(t)y'_1(t) + u'_2(t)y'_2(t) = g(t).$$

**Problem 13.1:** Using Variation of Parameters, find a particular solution of

$$y''(t) - y'(t) - 2y(t) = 2e^{-t}.$$

**Solution of 13.1:** The characteristic equation of the homog. version is $r^2 - r - 2 = 0$. Factoring $(r - 2)(r + 1) = 0$, whose roots are $r = 2, r = -1$, so

$$y_1(t) = e^{-t}, \quad y_2(t) = e^{2t}.$$
We also need the derivatives

\[ y'_1(t) = -e^{-t}, \quad y'_2(t) = 2e^{2t}. \]

The function \( g(t) \) is the right hand side (after we have divided by the coefficient of \( y''(t) \), in this case it is 1), so \( g(t) = 2e^{-t}. \)

We are looking for two functions \( u'_1(t) \) and \( u'_2(t) \) such that

\[ u'_1(t)e^{-t} + u'_2(t)e^{2t} = 0, \]
\[ u'_1(t)(-e^{-t}) + u'_2(t)(2e^{2t}) = 2e^{-t}. \]

Cleaning up (multiplying by \( e^t \))

\[ u'_1(t) + u'_2(t)e^{3t} = 0, \]
\[ -u'_1(t) + 2u'_2(t)e^{3t} = 2. \]

From the first equation, we get

\[ u'_1(t) = -e^{3t}u'_2(t). \]

Plugging into the second

\[ e^{3t}u'_2(t) + 2e^{3t}u'_2(t) = 2. \]

Collecting terms

\[ 3e^{3t}u'_2(t) = 2. \]

Dividing by \( 3e^{3t} \):

\[ u'_2(t) = \frac{2}{3}e^{-3t}. \]

Going back to \( u'_1(t) \):

\[ u'_1(t) = -e^{3t}\frac{2}{3}e^{-3t} = -\frac{2}{3}. \]

So we have

\[ u'_1(t) = -\frac{2}{3}, \quad u'_2(t) = \frac{2}{3}e^{-3t}. \]

Integrating (we don’t have to worry about the \(+C\) )

\[ u_1(t) = -\frac{2}{3}t, \quad u_2(t) = \frac{2}{9}e^{-3t}. \]

Finally, we plug these into

\[ Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t). \]
So

\[ Y(t) = (-\frac{2}{3})e^{-t} - \frac{2}{9}e^{-3t}e^{2t} = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-t}. \]

**First Answer to 13.1:** A particular solution is \( Y(t) = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-t}. \)

But since the second term is a multiple of \( y_1(t) \) and adding or subtracting any constant multiple of \( y_1(t) \) and/or \( y_2(t) \) from a particular solution is still (another, possibly simpler) particular solution, we can forget about the second term and get

**Second Answer to 13.1:** An even better particular solution is \( Y(t) = -\frac{2}{3}te^{-t}. \)

**Problem 13.2:** Using Variation of Parameters, find a particular solution of

\[ y''(t) - 2y'(t) + y(t) = e^t + t^2. \]

**Solution of 13.2:** The characteristic equation of the homog. version is \( r^2 - 2r + 1 = 0. \) Factoring \((r - 1)^2 = 0,\) and there is a **double root**, \( r = 1. \) So

\[ y_1(t) = e^t, \quad y_2(t) = te^t. \]

We also need the derivatives

\[ y_1'(t) = e^t, \quad y_2'(t) = (t + 1)e^t. \]

The function \( g(t) \) is the **right hand side** (after we have divided by the coefficient of \( y''(t), \) in this case it is 1), so \( g(t) = \frac{e^t}{1+t^2}. \)

We are looking for two functions \( u_1'(t) \) and \( u_2'(t) \) such that

\[ u_1'(t)e^t + u_2'(t)(t+1)e^t = 0, \]

\[ u_1'(t)e^t + u_2'(t)te^t = \frac{e^t}{1+t^2}. \]

Cleaning up (dividing by \( e^t \))

\[ u_1'(t) + u_2'(t)t = 0, \]

\[ u_1'(t) + (t+1)u_2'(t) = \frac{1}{1+t^2}. \]

From the first equation, we get

\[ u_1'(t) = -tu_2'(t). \]
Plugging into the second
\[-tu'_2(t) + (t + 1)u'_2(t) = \frac{1}{1 + t^2},\]
Simplifying:
\[u'_2(t) = \frac{1}{1 + t^2}.\]
Going back to \(u'_1(t)\):
\[u'_1(t) = -tu'_2(t) = -\frac{t}{1 + t^2}\]
So we have
\[u'_1(t) = -\frac{t}{1 + t^2}, \quad u'_2(t) = \frac{1}{1 + t^2}.\]
Integrating (we don’t have to worry about the \(+C\))
\[u_1(t) = -\frac{1}{2} \ln(1 + t^2), \quad u_2(t) = \arctan t.\]
Finally, we plug these into
\[Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)\]
So
\[Y(t) = -\frac{1}{2} e^t \ln(1 + t^2) + te^t \arctan t.\]

**Answer to 13.2:** A particular solution is \(Y(t) = -\frac{1}{2} e^t \ln(1 + t^2) + te^t \arctan t.\)

**Problem 13.3:** Verify that the given functions \(y_1(x)\), \(y_2(x)\) are solutions of the corresponding homogeneous linear diff.eq., and find the general solution of the diff.eq.
\[x^2y''(x) - 3xy'(x) + 4y(x) = x^2 \ln x, \quad x > 0; \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x.\]

**Solution of 13.3:** \(y_1(x) = x^2, \quad y'_1(x) = 2x, \quad y''_1(x) = 2,\) so
\[x^2y''_1(x) - 3xy'_1(x) + 4y_1(x) = x^2(2) - 3x(2x) + 4x^2 = 2x^2 - 6x^2 + 4x^2 = 0.\]

Also
\[y_2(x) = x^2 \ln x, \quad y'_2(x) = 2x \ln x + x, \quad y''_2(x) = 2 \ln x + 2 + 1 = 2 \ln x + 3,\] so
\[x^2y''_2(x) - 3xy'_2(x) + 4y_2(x) = x^2(2 \ln x + 3) - 3x(2x \ln x + x) + 4x^2 \ln x = 0.\]
So both \(y_1(x) = x^2\) and \(y_2(x) = x^2 \ln x\) are indeed solutions of the homogeneous version.

The function \(g(x)\) is the right hand side after we have divided by the coefficient of \(y''(t)\), so \(g(x) = \ln x\).
We are looking for two functions $u_1'(x)$ and $u_2'(x)$ such that

$$u_1'(x)x^2 + u_2'(x)x^2 \ln x = 0,$$
$$u_1'(x)(2x) + u_2'(x)(2x \ln x + x) = \ln x,$$

From the first equation

$$u_1'(x) = -(\ln x)u_2'(x).$$

Plugging into the second

$$-(\ln x u_2'(x))(2x) + u_2'(x)(2x \ln x + x) = \ln x,$$

Simplifying:

$$u_2'(x) = x^{-1} \ln x.$$

Going back to $u_1'(x)$:

$$u_1'(x) = - (\ln x)^2 x^{-1}.$$

So we have

$$u_1'(x) = - (\ln x)^2 x^{-1}, \quad u_2'(x) = (\ln x)x^{-1}.$$

Integrating (we don’t have to worry about the $+C$)

$$u_1(x) = -\frac{1}{3}(\ln x)^3, \quad u_2(x) = \frac{1}{2}(\ln x)^2.$$

Finally, we plug these into

$$Y(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

So a particular solution is

$$Y(x) = -\frac{1}{3}(\ln x)^3(x^2) + \frac{1}{2}(\ln x)^2(x^2 \ln x) = \left(\frac{1}{2} - \frac{1}{3}\right)x^2(\ln x)^3 = \frac{1}{6}x^2(\ln x)^3.$$

So, a particular solution is $Y(x) = \frac{1}{6}x^2(\ln x)^3$.

Finally, to get the general solution of the diff.eq. we add the general solution of the homogeneous version $c_1y_1(x) + c_2y_2(x)$, which in this problem is $c_1x^2 + c_2x^2 \ln x$.

Answer to 13.3: The general solution of the diff.eq. is $y(x) = c_1x^2 + c_2x^2 \ln x + \frac{1}{6}x^2(\ln x)^3$. 

5