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In order to find the general solution of a (general) linear second-order diff.eq.
\[ y''(t) + p(t) y'(t) + q(t) y(t) = g(t) \]
you find the general solution of the corresponding homogeneous version (obtained by replacing \( g(t) \) by 0)
\[ y''(t) + p(t) y'(t) + q(t) y(t) = 0 \]
that has the format \( c_1 y_1(t) + c_2 y_2(t) \) for some specific solutions \( y_1(t), y_2(t) \) (whose Wronskian is not identically zero).

Then by **hook or crook** you find ONE specific solution \( P(t) \) (called the particular solution) of the original (inhomogeneous diff.eq.). Then
\[
\text{General Solution} = c_1 y_1(t) + c_2 y_2(t) + P(t).
\]

In other words


**How to find Particular Solutions of Constant Coefficient Inhomog. linear diff.eqs.**

If the right side is a (specific) polynomial of degree \( k \), your first try is a generic polynomial (with undetermined coefficients) of degree \( k \). In particular if it is a constant, you try out \( g(t) = A_0 \).

If the right side is a (specific) exponential, \( e^{\alpha t} \), your first try is \( A_0 e^{\alpha t} \).

If the right side is a (specific) exponential times trig, \( e^{\alpha t} \sin \beta t \), or \( e^{\alpha t} \cos \beta t \), or, more generally, \( e^{\alpha t}(A \sin \beta t + B \cos \beta t) \) your first try is \( A e^{\alpha t}(A_0 \sin \beta t + B_0 \cos \beta t) \).

If the right side is a (specific) exponential, times a specific polynomial of degree \( k \), \( e^{\alpha t} p(t) \), your first try is \( A e^{\alpha t}(A_0 t^k + A_1 t^{k-1} + \ldots + A_k t^0) \).

If the first try does not work, multiply the proposed Particular Solution by \( t \).

**Problem 12.1**: Find a particular solution of the diff.eq.
\[ y''(t) + 3 y'(t) + 2 y(t) = 6 \]

**Solution of 12.1**: Since the right side is a constant, you try out
\[ y(t) = A_0 \]
where $A_0$ is a constant yet to be determined.

Now $y'(t) = 0$, $y''(t) = 0$ so, plugging into the diff.eq.

$$0 + 3 \cdot 0 + 2 \cdot A_0 = 6$$

Solving the equation $2A_0 = 6$ we get $A_0 = 3$. So a particular solution is $y(t) = 3$.

**Ans. to 12.1:** a particular solution is $y(t) = 3$.

**Problem 12.2:** Find a particular solution of the diff.eq.

$$y''(t) + 3y'(t) + 2y(t) = 2t^2 + 8t + 7$$

**Solution of 12.2:** Since the right side is a polynomial in $t$ of degree 2, you try out

$$y(t) = A_0t^2 + A_1t + A_2$$

where $A_0, A_1, A_2$ are constants yet to be determined.

Now $y'(t) = 2A_0t + A_1, y''(t) = 2A_0$ so, plugging into the diff.eq.

$$2A_0 + 3(2A_0t + A_1) + 2(A_0t^2 + A_1t + A_2) = 2t^2 + 8t + 7$$

Simplifying:

$$(2A_0)t^2 + (2A_1 + 6A_0)t + (2A_0 + 3A_1 + 2A_2) = 2t^2 + 8t + 7$$

We now compare coefficients of $t^2, t, 0$ on both sides getting three linear equations for the three unknowns $A_0, A_1, A_2$.

$$2A_0 = 2 \quad , \quad 2A_1 + 6A_0 = 8 \quad , \quad 2A_0 + 3A_1 + 2A_2 = 7$$

Solving these equation, we get from the first, $A_0 = 1$, then from the second $2A_1 + 6 \cdot 1 = 8$, so $A_1 = 1$, and from the from the third $2(1) + 3(1) + 2A_2 = 7$, so $2A_2 = 2$ and $A_2 = 1$.

Going back to the general template $A_0t^2 + A_1t + A_2$ we get that a particular solution is $y(t) = t^2 + t + 1$.

**Ans. to 12.2:** a particular solution is $y(t) = t^2 + t + 1$.

**Problem 12.3:** Find a particular solution of the diff.eq.

$$y''(t) + 3y'(t) + 2y(t) = e^{2t}$$

**Solution of 12.3:** Since the right side is a pure exponential function $e^{2t}$, we simply try

$$y(t) = A_0e^{2t}$$
where $A_0$ is a constant yet \textbf{to be determined}.

Now $y'(t) = 2A_0e^{2t}$, $y''(t) = 4A_0e^{2t}$ so, plugging into the diff.eq.

$$4A_0e^{2t} + 3(2A_0e^{2t}) + 2(A_0e^{2t}) = e^{2t}$$

Simplifying:

$$(4A_0 + 6A_0 + 2A_0)e^{2t} = e^{2t}$$

$$12A_0e^{2t} = e^{2t}$$

Dividing by $e^{2t}$, $12A_0 = 1$, and solving for $A_0$, we get $A_0 = \frac{1}{12}$.

\textbf{Ans. to 12.3:} a particular solution is $y(t) = \frac{1}{12}e^{2t}$.

**Problem 12.4:** Find a particular solution of the diff.eq.

$$y''(t) + 3y'(t) + 2y(t) = (12t + 7)e^{2t}$$

\textbf{Solution of 12.4:} Since the right side is a pure \textit{exponential function} $e^{2t}$, \textit{times} a certain polynomial of degree 1, we try

$$y(t) = (A_0t + A_1)e^{2t}$$

where $A_0, A_1$ are constants yet \textbf{to be determined}.

Now $y'(t) = (2A_0t + (2A_1 + A_0))e^{2t}$, $y''(t) = ((4A_0t + (4A_1 + 4A_0))e^{2t}$, so, plugging into the diff.eq.

$$((4A_0t + (4A_1 + 4A_0))e^{2t} + 3((2A_0t + (2A_1 + A_0)) + 2(A_0t + A_1)e^{2t} = (6t + 11)e^{2t}$$

Simplifying:

$$(12A_0t + (12A_1 + 7A_0))e^{2t} = (12t + 7)e^{2t}$$

Dividing by $e^{2t}$:

$$12A_0t + (12A_1 + 7A_0) = 12t + 7$$

Comparing coefficients:

$$12A_0 = 12, \quad 12A_1 + 7A_0 = 7$$

From the first equation: $A_0 = 1$ from the second $12A_1 + 7 = 7$, so $A_1 = 0$.

Going back to the general template $(A_0t + A_1)e^{2t}$ we get that a particular solution is $y(t) = te^{2t}$.

\textbf{Ans. to 12.4:} a particular solution is $y(t) = te^{2t}$.

**Problem 12.5:** Find a particular solution of the diff.eq.

$$y''(t) + 3y'(t) + 2y(t) = e^{-t}$$
Solution of 12.5: Since the right side is a pure exponential function $e^{-t}$, the first try is

$$y(t) = A_0 e^{-t}.$$  

Now

$$y'(t) = -A_0 e^{-t}, \quad y''(t) = A_0 e^{-t}.$$  

Plugging into the diff.eq.

$$A_0 e^{-t} + 3(-A_0 e^{-t}) + 2A_0 e^{-2t} = e^{-t}$$  

Simplifying, we get:

$$0 = e^{-t}.$$  

NONSENSE! So the first try failed. We now have a second try, by multiplying the first try by $t$.

$$y(t) = A_0 t e^{-t}.$$  

Now

$$y'(t) = A_0 (1 - t) e^{-t}, \quad y''(t) = A_0 (t - 2) e^{-t}.$$  

Plugging into the diff.eq.

$$A_0 (t - 2)e^{-t} + 3(1 - t)A_0 e^{-t} + 2A_0 e^{-2t} = e^{-t}$$  

Simplifying

$$A_0 (t - 2) + 3(1 - t) + 2A_0 = e^{-t}$$  

$$A_0 (t - 2 + 3 - 3t + 2t) = e^{-t}$$  

$$A_0 e^{-t} = e^{-t}$$  

Dividing by $e^{-2t}$, we get

$$A_0 = 1.$$  

Going back to the template, we get that a particular solution is $y(t) = te^{-t}$.

Problem 12.1': Find the general solution of the diff.eq.

$$y''(t) + 3y'(t) + 2y(t) = 6$$  

Solution of 12.1: We first find the general solution of the homogeneous version

$$y''(t) + 3y'(t) + 2y(t) = 0$$  

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The characteristic equation is \( r^2 + 3r + 2 = 0 \), factorizing \((r + 1)(r + 2) = 0\), so the roots are \( r_1 = -1 \), \( r_2 = -2 \), and the general solution of the homog. version is

\[ y(t) = c_1 e^{-t} + c_2 e^{-2t} \].

We now do the Particular Solution (see Problem 12.1), and get P.S. \( y(t) = 3 \). We now Add them up and get that the general solution of our diff.eq. is:

\[ y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t} \].

**Ans. to 12.1**: \( y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t} \)

**Problem 12.5**: Find the general solution of the diff.eq.

\[ y''(t) + 3y'(t) + 2y(t) = e^{-t} \]

**Solution of 12.5**: We first find the general solution of the homogeneous version

\[ y''(t) + 3y'(t) + 2y(t) = 0 \]

The characteristic equation is \( r^2 + 3r + 2 = 0 \), factorizing \((r + 1)(r + 2) = 0\), so the roots are \( r_1 = -1 \), \( r_2 = -2 \), and the general solution of the homog. version is

\[ y(t) = c_1 e^{-t} + c_2 e^{-2t} \].

We now do the Particular Solution (see Problem 12.5), and get P.S. \( y(t) = te^{-t} \). We now Add them up and get that the general solution of our diff.eq. is:

\[ y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t} \].

**Ans. to 12.5**: \( y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t} \)

**Problem 12.1**: Solve the initial value problem

\[ y''(t) + 3y'(t) + 2y(t) = 6 \quad , \quad y(0) = 6 \quad , \quad y'(0) = -4 \]

**Solution of 12.1**: We first do 12.1’, getting the general solution

\[ y(t) = 3 + c_1 e^{-t} + c_2 e^{-2t} \].

Now

\[ y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \].

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So
\[ y(0) = 3 + c_1 e^{-0} + c_2 e^{-2 \cdot 0} = 3 + c_1 + c_2 \] .
\[ y'(0) = -c_1 e^{-0} - 2c_2 e^{-2 \cdot 0} = -c_1 - 2c_2 \]

Using our initial condition, we have to solve
\[ 3 + c_1 + c_2 = 6 , \quad -c_1 - 2c_2 = -4 \]
i.e.
\[ c_1 + c_2 = 3 , \quad c_1 + 2c_2 = 4 \]

Subtracting the second equation from the first, we get \( c_2 = 1 \) and plugging this into the first (or second) equation we get \( c_1 = 2 \). Going back to the general solution, we get the solution is
\[ y(t) = 3 + 2e^{-t} + e^{-2t} \]

\textbf{Ans. to 12.1\textsuperscript{”}: } y(t) = 3 + 2e^{-t} + e^{-2t}.

\textbf{Problem 12.5\textsuperscript{”}: } Solve the initial value problem
\[ y''(t) + 3y'(t) + 2y(t) = e^{-t} , \quad y(0) = 1 , \quad y'(0) = -1 \]

\textbf{Solution of 12.5\textsuperscript{”}: } We first do 12.5\textsuperscript{’}, getting the general solution
\[ y(t) = te^{-t} + c_1 e^{-t} + c_2 e^{-2t} \] .

Now
\[ y'(t) = (1-t)e^{-t} - c_1 e^{-t} - 2c_2 e^{-2t} \] .

Plugging-in \( t = 0 \):
\[ y(0) = 0 \cdot e^{-0} + c_1 e^{-0} + c_2 e^{-2 \cdot 0} = c_1 + c_2 \] .
\[ y'(0) = (1-0)e^{-0} - c_1 e^{-0} - 2c_2 e^{-2 \cdot 0} = 1 - c_1 - 2c_2 \] .

So
\[ c_1 + c_2 = 1 , \quad 1 - c_1 - 2c_2 = -1 \]

Simplifying
\[ c_1 + c_2 = 1 , \quad c_1 + 2c_2 = 2 \]

Subtracting the second equation from the first, we get \( c_2 = 1, \ c_1 = 0 \). Going back to the general solution we have
\[ y(t) = te^{-t} + 0 \cdot e^{-t} + 1 \cdot e^{-2t} = te^{-t} + e^{-2t} \] .

\textbf{Ans. to 12.5\textsuperscript{”}} y(t) = te^{-t} + e^{-2t}. 

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